Critical exponent in $\mathbb{H}^2 \times \mathbb{H}^2$

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Notations

- Let S be a fixed compact surface of genus $g \geq 2$.
- Let $\Gamma = \pi_1(S)$ be the fundamental group of S.
- Let $\mathcal{C}$ be the set of free homotopy classes of closed curves on S.
- If S is endowed with a marked hyperbolic metric, for every $c \in \mathcal{C}$, let $\ell(c)$ be the length of the (unique) geodesic representative of c on S.

One factor setting

Let S_0 be a compact surface of genus $g \geq 2$, S, with a marked hyperbolic metric. Equivalently, let ρ_0 be a faithful and discrete representation of $\Gamma = \pi_1(S)$ into $\mathrm{Isom}(\mathbb{H}^2)$ such that $S_0 = \mathbb{H}^2/\rho_0(\Gamma)$. Let $\gamma_0 = \rho_0(\gamma)$.

$$\begin{aligned}\delta(S_0) &:= \lim_{R\to\infty} \frac{1}{R} \log \#\{\gamma \in \Gamma \,|\, d(\rho_0(\gamma)o, o) \leq R\}\\ &= \lim_{R\to\infty} \frac{1}{R} \log \#\{c \in \mathcal{C} \,|\, \ell_0(c) \leq R\}\\ &= \lim_{R\to\infty} \frac{1}{R} \log \mathrm{Vol}\, B(o,R)\\ &= 1.\end{aligned}$$

Two factors setting

Let S_1, S_2 be two hyperbolic metrics on S. Equivalently, let ρ_1, ρ_2 be two faithful and discrete representations of Γ into $\mathrm{Isom}(\mathbb{H}^2)$ such that $S_i = \mathbb{H}^2/\rho_i(\Gamma)$. Let $\gamma_i = \rho_i(\gamma)$. Consider the diagonal action of Γ on $\mathbb{H}^2 \times \mathbb{H}^2$.

$$\begin{aligned}\delta(S_1,S_2) &:= \lim_{R\to\infty} \frac{\log \#\{\gamma \in \Gamma \,|\, d(\gamma_1 o, o) + d(\gamma_2 o, o) \leq R\}}{R}\\ &= \lim_{R\to\infty} \frac{1}{R} \log \#\{c \in \mathcal{C} \,|\, \ell_1(c) + \ell_2(c) \leq R\}\end{aligned}$$

It is not hard to see that

- δ is continuous
- $\delta \in (0, 1/2]$.
- $\delta(S_1, S_2)$ is invariant under the diagonal action of $MCG(S)$.

Comparison to Thurston distance

Let $S_1, S_2 \in \mathrm{Teich}(S)$. Let $\mathrm{dil}^- := \inf_{c\in C} \frac{\ell_2(c)}{\ell_1(c)}$ and $\mathrm{dil}^+ := \sup_{c\in C} \frac{\ell_2(c)}{\ell_1(c)}$. The Thurston distance is defined by :

$$d_T(S_1, S_2) := \log \max(\mathrm{dil}^+, 1/\mathrm{dil}^-).$$

If $\lim \delta(S_1, S_n) = 0$ then $\lim d_T(S_0, S_n) = +\infty$. The converse is false : modify only a subsurface.

Rigidity theorem [1]

C. Bishop and C. Steger proved the following :

$$\delta(S_1, S_2) = 1/2 \text{ if and only if } d_T(S_1, S_2) = 0.$$

Main results

- **Theorem** Let S_0 be a fixed marked hyperbolic surface. Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of marked hyperbolic surface. We then have the following equivalence

$$\lim \delta(S_1, S_n) = 1/2 \text{ if and only if } \lim d_T(S_1, S_n) = 0.$$

Corollary Let $(S_n)_{n\in\mathbb{N}}$ and $(S'_n)_{n\in\mathbb{N}}$ be two sequences of marked hyperbolic surfaces. Suppose that at least one of the sequences stays in the thick part of $\mathrm{Teich}(S)$. Then we have the equivalence

$$\lim_{n\to\infty} \delta(S_n, S'_n) = \frac{1}{2} \text{ if and only if } \lim_{n\to\infty} d_T(S_n, S'_n) = 0.$$

Optimality There exists two sequences of marked hyperbolic surfaces $(S_n)_{n\in\mathbb{N}}$ and $(S'_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \delta(S_n, S'_n) = \frac{1}{2} \text{ and } \lim d_T(S_n, S'_n) = +\infty.$$

Example for Theorem

Thanks to the rigidity theorem, we just have to study the asymptotic behaviour in $\partial\,\mathrm{Teich}(S)$. Along Dehn twists, this is easy. Let α be a simple closed curved and τ_α be the Dehn twist around α. Aim : show that $\delta(S_0, \tau_\alpha^{2n} S_0)$ does not tend to $1/2$ as $n \to \infty$.

- **Invariance under the MCG(S)**

$$\delta(S_0, \tau_\alpha^{2n} S_0) = \delta(\tau_\alpha^{-n} S_0, \tau_\alpha^{n} S_0).$$

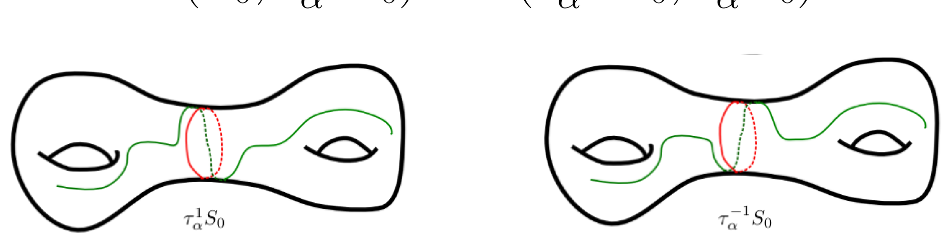

Figure 1: Left and right Dehn twists

- **Convexity of length along earthquake** [2]

$$\left(\ell_0(\tau_\alpha^n c) + \ell_0(\tau_\alpha^{-n} c)\right)_{n\in\mathbb{N}}, \text{ is increasing}$$

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Example for Optimality

Along pinching the Thurston distance tends to infinity but the Weil-Peterson distance is bounded. Let α a simple closed curve on S. Let S_t the surface we get by pinching α by a factor e^{-t}. We have $d_T(S_t, S_{t+1}) \geq 1$ (compare the length of α). We can show that $\lim \delta(S_t, S_{t+1}) = 0$.

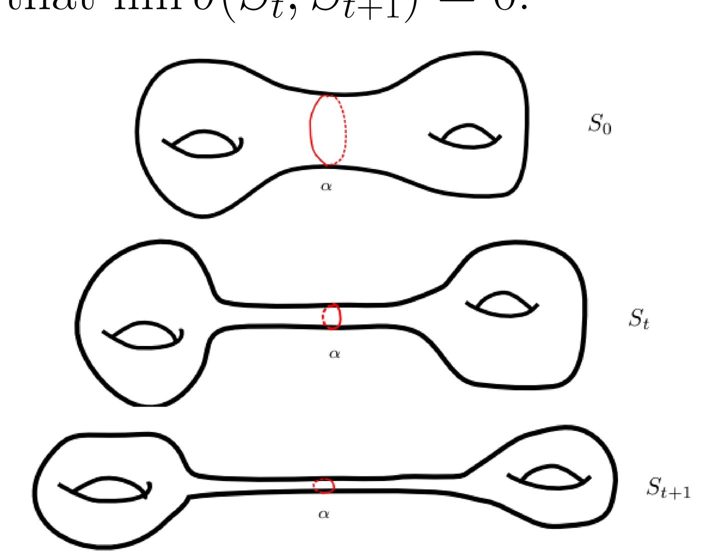

Figure 2: Pinching along simple closed curve

Question

Let $\mathcal{L}$ a measured geodesic lamination. This define an earthquake path $t \mapsto S_t = EQ^t_{\mathcal{L}}(S_0)$. Does $\lim_{t\to\infty} \delta(S_0, S_t)$ exists ? Can we give a simple expression in terms of $\mathcal{L}$?

Anti de Sitter interpretation

Globally hyperbolic (GH) is a kind of AdS-manifolds, corresponding to *quasi-Fuchsian* in Riemannian setting. They are parametrized by $\mathrm{Teich}(S) \times \mathrm{Teich}(S)$. A closed geodesic on a GH manifold parametrized by (S_1, S_2), correspond to a pair of geodesic on (S_1, S_2). Its Lorentzian length is the mean $\frac{l_1(c)+l_2(c)}{2}$.

Consequences Our theorem and examples are analogue of known results in quasi-Fuchsian manifolds. Example in quasi-Fuchsian setting are treated by C. McMullen in [3] and the isolation theorem is due to A. Sanders [4].

[1] Christopher Bishop and Tim Steger.
Three rigidity criteria for psl(2,r).
Bulletin of the American Mathematical Society, 24(1):117–123, 1991.

[2] Scott Wolpert.
Geodesic length functions and the nielsen problem.
Journal of Differential Geometry, 25(2):275–296, 1987.

[3] Curtis T McMullen.
Hausdorff dimension and conformal dynamics i: Strong convergence of kleinian groups.
1999.

[4] Andrew Sanders.
Entropy, minimal surfaces, and negatively curved manifolds.
arXiv preprint arXiv:1404.1105, 2014.

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