Entropy of embedded surfaces in quasi-fuchsian manifolds

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Abstract

We compare critical exponent for quasi-Fuchsian groups acting on the hyperbolic 3-space and entropy of invariant disks embedded in $\mathbb{H}^3$. We give a rigidity theorem for all embedded surfaces when the action is Fuchsian and a rigidity theorem for negatively curved surfaces when the action is quasi-Fuchsian.

1   Introduction

The aim of this paper is to compare two geometric invariants of Riemannian manifolds : critical exponent and volume entropy. The first one is defined through the action of the fundamental group on the universal cover, the second one is defined for compact manifolds as the exponential growth rate of the volume of balls in the universal cover. These two invariants have been studied in many cases, we pursue this study for quasi-Fuchsian manifolds.

Let Γ be a group acting on a simply connected Riemannian manifold (X, g). If the action on X is discrete we define the *critical exponent* by

$$\delta(\Gamma) := \limsup_{R \to \infty} \frac{1}{R} \operatorname{Card}\{\gamma \in \Gamma \mid d(\gamma \cdot o, o) \leq R\}, \tag{1}$$

where o is any point in X. It does not depends on this particular base point thanks to triangular inequality. If we want to insist on the space on which Γ acts we will write $\delta(\Gamma, X)$.

The volume entropy $h(g)$ of a Riemannian compact manifold (Σ, g) is defined by

$$h(g) := \lim_{R \to \infty} \frac{\log \operatorname{Vol}_g(B_g(o, R))}{R}, \tag{2}$$

where $B_g(o, R)$ be the ball of radius R and center o in the universal cover of Σ. We will also use the notation $h(X)$ for simply connected manifolds X as the exponential growth rate of its balls.

Volume entropy coincides with the critical exponent of $\pi_1(\Sigma)$ acting on $\widetilde{\Sigma}$.

Moreover, a famous theorem of G. Besson, G. Courtois and S. Gallot [BCG95] said that the entropy allows to distinguish hyperbolic metric in the set of all metrics, $\operatorname{Met}(\Sigma)$. Remark that entropy is sensitive to homothetic transformations : for any $\lambda > 0$ we have $h(\lambda^2 g) = \frac{1}{\lambda} h(g)$. Assume that Σ admits an hyperbolic metric g_0 and let $\operatorname{Met}_0(\Sigma)$ be the set of metrics on Σ whose volume is equal to $\operatorname{Vol}(\Sigma, g_0)$, then Besson, Courtois, Gallot's theorem says for all $g \in \operatorname{Met}_0(\Sigma)$:

$$h(g) \geq h(g_0). \tag{3}$$

with equality if and only if $g = g_0$.

Our aim is to study the behavior of the volume entropy for a subset of all the metrics on a surface. This subset is the metrics induced by an incompressible embedding into a quasi-Fuchsian manifolds. It has not the cone structure of $\mathrm{Met}(\Sigma)$: it is not invariant by all homothetic transformations. Hence we will look at the behavior of $h(g)$ without normalization by the volume.

Let S be a compact surface of genus $g \geq 2$ and $\Gamma = \pi_1(S)$ its fundamental group. A Fuchsian representation of Γ is a faithful and discrete representation in $\mathrm{PSL}_2(\mathbb{R})$. A quasi-Fuchsian representation is a perturbation of Fuchsian representation in $\mathrm{PSL}_2(\mathbb{C})$. More precisely it is a discrete and faithfull representations of Γ into $\mathrm{Isom}(\mathbb{H}^3)$, such that the limit set on $\partial \mathbb{H}^3$ is a Jordan curve. A celebrated theorem of R. Bowen [Bow79], asserts that for quasi-Fuchsian representations, critical exponent is minimal and equal to 1 if and only if the representation is Fuchsian.

We choose an isometric, totally geodesic embedding of $\mathbb{H}^2$ in $\mathbb{H}^3$ (The equatorial plane in the ball model for example). This embedding gives a inclusion $i : \mathrm{Isom}(\mathbb{H}^2) \to \mathrm{Isom}(\mathbb{H}^3)$.

Let ρ be a Fuchsian representation of Γ. The group Γ acts naturally on $\mathbb{H}^2$, respectively $\mathbb{H}^3$, by ρ, respectively $i \circ \rho$. For every points $o \in \mathbb{H}^2$ we have

$$d_{\mathbb{H}^3}(i \circ \rho(\gamma)o, o) = d_{\mathbb{H}^2}(\rho(\gamma)o, o),$$

since $\mathbb{H}^2$ is totally geodesic in $\mathbb{H}^3$. The critical exponent for these two actions of Γ are then equal

$$\delta(\Gamma, \mathbb{H}^3) = \delta(\Gamma, \mathbb{H}^2) = 1.$$

In light of this trivial example, two questions rise up. What is the entropy of a Γ invariant disk which is not totally geodesic ? What happens when we modify the Fuchsian representation in $\mathrm{PSL}_2(\mathbb{C})$?

We will answer to the first question. Since ρ is a Fuchsian representation, the critical exponent of Γ acting on $\mathbb{H}^3$ through $i \circ \rho$ is 1, and we have the following

Theorem 1.1. *Let Σ be a Γ invariant disk embedded in $\mathbb{H}^3$. We have*

$$h(\Sigma) \leq \delta(\Gamma, \mathbb{H}^3), \tag{4}$$

equality occurs if and only if Σ is the totally geodesic hyperbolic plane preserved by Γ.

Remarks that $\delta(\Gamma, \mathbb{H}^3) = h(\Sigma, g_0)$, hence the last theorem can be rewritten as follow :

Theorem 1.2. *For all metrics g obtained as induced metrics by an incompressible embedding in a Fuchsian manifold we have*

$$h(g) \leq h(g_0) \tag{5}$$

with equality if and only if $g = g_0$.

We *did not* renormalize by the volume, this explains the dichotomy between (3) and (5).

We will prove this theorem in the next section. The inequality is trivial since the induce distance between two points is always greater than the distance in $\mathbb{H}^3$: $d_\Sigma \geq d_{\mathbb{H}^3}$, but the rigidity is not. We have no geometrical (curvature) hypothesis on Σ, therefore it is not obvious at all to show that the inequality is strict as soon as Σ is not totally geodesic. Indeed we cannot use the "usual" techniques of negative curvature like Bowen-Margulis measure, or even the uniqueness of geodesic between two points.

We obtain an answer to the second question under a geometrical hypothesis on the curvature :

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Theorem 1.3. *Let Γ be a quasi-Fuchsian group and $\Sigma \subset \mathbb{H}^3$ a $\Gamma-$ invariant embedded disk. We suppose that Σ endowed with the induced metric has negative curvature. We then have*

$$h(\Sigma) \leq I(\Sigma, \mathbb{H}^3)\delta(\Gamma, \mathbb{H}^3),$$

where $I(\Sigma, \mathbb{H}^3)$ is the geodesic intersection between Σ and $\mathbb{H}^3$. Moreover the inequality occurs if and only if the length spectrum of Σ/Γ is proportional to the one of $\mathbb{H}^3/\Gamma$.

The *geodesic intersection* will be defined in section 3.1. Roughly it is the average ratio of the length between two points of Σ for the extrinsic and intrinsic distance. We need the curvature assumption to define and use this invariant.

As a corollary of Theorem 1.3, we obtain

Corollary 1.4. *Under the assumptions of Theorem 1.3 we have*

$$h(\Sigma) \leq \delta(\Gamma, \mathbb{H}^3),$$

with equality if and only if Γ is fuchsian and Σ is the totally geodesic hyperbolic plane, preserved by Γ.

Theorem 1.3 has to be compared to results obtained by G. Knieper who compared entropy for two different metrics on the same manifolds and our proof of Theorem 1.3 follows his paper [Kni95].

It is also related to the work of M. Bridgeman and E. Taylor [BT00], indeed we answer by the negative to Question 2 of their paper. Finally, U. Hamenstadt looked in [Ham02] at the geodesic intersection between the boundary of convex hull and $\mathbb{H}^3$ for quasi-Fuchsian manifolds.

As we said, the two proofs are very different one from each others. For the Fuchsian case, we give precise estimates for the length of some paths of the hyperbolic plane. We show that in some sense the length between two points on Σ is much greater than the extrinsic distance between those two points. For quasi-Fuchsian manifolds, we use well known techniques of negative curvature geometry : we compare the Patterson Sullivan measures for $\mathbb{H}^3$ and for Σ.

2 Fuchsian case

In this section we are going to prove Theorem 1.1. This theorem has a strong condition on Γ, ie. it is conjugate to a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ but we make no geometrical assumptions on Σ. As we said, there could be more than one geodesic between two points on Σ.

We already remarked that the inequality is trivial, as is the equality when Σ is totally geodesic. Therefore, the only thing left to prove is the strict inequality when Σ is not totally geodesic or in other words if $\Sigma \neq \mathbb{H}^2$ then $h(\Sigma) < 1$.

The proof of the theorem is based on the comparison between the distances on equidistant surfaces of the totally geodesic Γ-invariant hyperbolic plane. We are going to prove several lemmas which together gives Theorem 1.1. The strict inequality follows directly from 2.2 and 2.8. We denote by $\mathbb{D}$ the totally geodesic, Γ-invariant plane. The induced metric on $\mathbb{D}$ is the usual hyperbolic metric, and we will denote it by $\mathbb{H}^2$. We are first going to see that between all the equidistant surfaces, $\mathbb{H}^2$ has the biggest entropy. Then we will make this argument work when only one part of the surface is "above" $\mathbb{D}$. The idea to prove it, is to consider another distance d_m on $\mathbb{D}$, which will be used as an intermediary between Σ and $\mathbb{H}^2$. We will explain, after the definition of d_m how the two comparisons will be proved.

Let us begin to parametrize $\mathbb{H}^3$ by $\mathbb{H}^2 \times \mathbb{R}$ as follows : take an orientation for the unit normal tangent space of $\mathbb{H}^2$, then to a point $x \in \mathbb{H}^3$ we associate orthogonal projection on $\mathbb{H}^2$ thi is

the first parameter of the parametrization. The oriented distance along this geodesic gives the second one. We call s the projection from $\mathbb{H}^3$ to $\mathbb{H}^2$. Hence the parametrisation, called Fermi coordinates, is the defined by

$$
\begin{array}{ccc}
\mathbb{H}^3 & \mapsto & \mathbb{H}^2 \times \mathbb{R} \\
z & \to & (s(z), \hat{d}(z, s(z)))
\end{array}
$$

where the $\hat{d}$ is the oriented distance by the choice of the orientation on the unit normal tangent of $\mathbb{H}^2$. With this parametrization, the metric on $\mathbb{H}^3$ is

$$ g_{\mathbb{H}^3} = \cosh^2(r)g_0 + dr^2. $$

Look at $S(r)$ the equidistant disk at distance r of $\mathbb{H}^2$, its metric, induces by the one on $\mathbb{H}^3$ is $g_r = \cosh^2(r)g_0$. It is isometric to a hyperbolic plane of curvature $\frac{1}{\cosh(r)}$, and its volume entropy is $h(S(r)) = \frac{h(0)}{\cosh(r)} = \frac{1}{\cosh(r)}$, hence the entropy is maximal if and only if $r = 0$. For the general case, we are going to refined this argument showing that it is sufficient that a small part of Σ is over $\mathbb{H}^2$ for the entropy to be strictly less than 1.

Let Σ be a embedded Γ-invariant disk in $\mathbb{H}^3$. We assume that $\Sigma \neq \mathbb{D}$, and we endowed Σ with its induced metric. Let x, y be two points on Σ. Let c_Σ be a geodesic on Σ linking x to y. We parametrize c_Σ by its Fermi coordinates, (c, r). We then have

$$
\begin{aligned}
d_\Sigma(x, y) &= \int_0^L \|c'_\Sigma(t)\|_\Sigma dt \\
&= \int_0^L \sqrt{r'(t)^2 + \cosh^2(r(t))\|c'(t)\|_{g_0}^2} \, dt. \\
&\geq \int_0^L \cosh(r(t))\|c'(t)\|_{g_0} dt. \tag{6}
\end{aligned}
$$

We now endowed $\mathbb{D}$ with another distance than the one coming from hyperbolic metric. It will play the role of intermediary to compare $d_\Sigma(x, y)$ on Σ with $d_{g_0}(s(x), s(y))$ on $\mathbb{H}^2$.

We call σ the restriction of s on Σ. Since $\Sigma \neq \mathbb{D}$, there exists $x_0 \in \mathbb{D} \setminus \Sigma$, $\varepsilon > 0$ and $\eta > 0$ such that

$$ d_{\mathbb{H}^3}(\sigma^{-1}B(x_0, \varepsilon), \mathbb{D}) > \eta. $$

This means that *all* the points in the pre-image of $B(x_0, \varepsilon)$ by σ are at distance greater than η from $\mathbb{D}$. We will assume that ε is smaller than the injectivity radius of $\mathbb{H}^2/\Gamma$ in order that the translations of $B(x_0, \varepsilon)$ by Γ are disjoint.

We now consider on $\mathbb{D}$ the metric g_m defined by putting weight on the translations of $B(x_0, \varepsilon)$ by Γ.

Definition 2.1. *We define g_m by*

$$ g_m := \cosh(\eta)^2 g_0, $$

on $\Gamma \cdot B(x_0, \varepsilon)$. and

$$ g_m := g_0, $$

elsewhere.

We will index by m objects which depends on this metric. Remark that this metric is not continuous but it still defined a length space. Let $c : [0, 1] \to \mathbb{D}$ be a C^1 path we then have

$$ \ell_m(c) = \int_0^1 \|\dot{c}(t)\|_{g_m} dt. $$

This gives a distance d_m on $\mathbb{D}$ by choosing :

$$d_m(x, y) := \inf_c \{\ell_m(c) \,|\, c(0) = x,\, c(1) = y\}.$$

In order to prove Theorem 1.1 we will compare the entropy of $(\mathbb{D}, d_m)$ with the one of Σ and the one of $\mathbb{H}^2$. The comparison whit the entropy of Σ is quite easy and follows quickly from the definition of d_m and the inequality (6). The comparison with the entropy of $\mathbb{H}^2$ is more subtle. Indeed, there exists geodesic of $\mathbb{H}^2$ which are geodesic for $(\mathbb{D}, d_m)$ (any lift of a closed geodesic which does not cross the ball $B(x_0, \varepsilon)/\Gamma$ on $\mathbb{H}^2/\Gamma$). We will first prove that two points of $\mathbb{D}$ which are joined by a geodesic of $\mathbb{H}^2$ which crosses often $\Gamma \cdot B(x_0, \epsilon)$ are much farther away from each other for d_m distance, cf Lemme 2.4. Then, we will use a large deviation theorem for the geodesic flow (Theorem 2.6), to show that there are few geodesic which does not cross $\Gamma \cdot B(x_0, \epsilon)$ (Lemme 2.7). It will follow from these two results that the balls of radius R for d_m are almost completely included in balls of radius R/C of $\mathbb{H}^2$ for $C > 1$ (Lemma 2.8). The two comparisons give the proof of Theorem 1.1.

The comparison between $h(\Sigma)$ and the critical exponent of $(\mathbb{D}, d_m)$ follows from the inequality 6 and the definition of d_m

Lemma 2.2. *We have*
$$h(\Sigma) \le \delta((\mathbb{D}, d_m)).$$

Proof. Let $x \in \Sigma$ and $o = \sigma(x) \in \mathbb{D}$. Since Σ/Γ is compact, we have

$$h(\Sigma) = \lim_{R \to \infty} \frac{\log \mathrm{Card}\{\gamma \in \Gamma \,|\, d_\Sigma(\gamma x, x) \le R\}}{R}.$$

And by definition

$$\delta((\mathbb{D}, d_m)) = \lim_{R \to \infty} \frac{\log \mathrm{Card}\{\gamma \in \Gamma \,|\, d_m(\gamma o, o) \le R\}}{R}.$$

It is sufficient to prove that $d_\Sigma(x, y) \ge d_m(s(x), s(y))$, for all $x, y \in \Sigma$. Let $c_\Sigma = (c, r)$ be a geodesic on Σ joining x to y. Recall that we have

$$d_\Sigma(x, y) \;\ge\; \int_0^L \cosh(r(t)) \|c'(t)\|_{g_0} dt.$$

If $c(t) \notin \Gamma \cdot B(x_0, \epsilon)$, then $\|c'(t)\|_{g_m} = \|c'(t)\|_{g_0}$. In particular

$$\|c'(t)\|_{g_m} \le \cosh(r(t)) \|c'(t)\|_{g_0}.$$

If $c(t) \in \Gamma \cdot B(x_0, \epsilon)$, then by definition of g_m, $\|c'(t)\|_{g_m} = \cosh(\eta)\|c'(t)\|_{g_0}$ and since Σ is "far" from $\mathbb{D}$, $r(t) > \eta$. In particular,

$$\|c'(t)\|_{g_m} \le \cosh(r(t)) \|c'(t)\|_{g_0}.$$

Finally

$$\begin{aligned} d_\Sigma(x, y) \;&\ge\; \int_0^L \|c'(t)\|_{g_m} dt \\ &\ge\; l_m(c) \\ &\ge\; d_m(s(x), s(y)) \end{aligned}$$

$\square$

Our next aim is to compare the distance d_m and $d_{\mathbb{H}^2}$. Let us fix some notations before stating the first lemma. For all $v \in T^1\mathbb{H}^2$, let ζ_R^v be the probability measure on $T^1\mathbb{H}^2$, defined for all borelian $E \subset T^1\mathbb{H}^2$ by :

$$\zeta_R^v(E) = \frac{1}{R} \int_0^R \chi_E \left(\phi_t^{\mathbb{H}^2}(v) \right) dt$$

where χ_E is the indicator function of E. For a borelian E which is a unitary tangent bundle of a subset of $\mathbb{D}$, $E := T^1A$, we have

$$\zeta_R^v(E) = \frac{1}{R} Leb\{t \in [0, R] | c_v(t) \in A\}$$

since $\phi_t^{\mathbb{H}^2}(v) \in E$ is equivalent to $c_v(t) = \pi\phi_t^{\mathbb{H}^2}(v) \in A$.

Let L be the Liouville measure on the unitary tangent bundle of the quotient surface $T^1\mathbb{H}^2/\Gamma$. We fix the following $K := T^1(\Gamma \cdot B(x_0, \varepsilon))$.

Definition 2.3. *Let $\kappa > 0$ be such $L(K/\Gamma) - 2\kappa > 0$. We define the following sets,*

$$\mathcal{E}(R) := \{v \in T^1\mathbb{H}^2 \mid |\zeta_R^v(K) - L(K/\Gamma)| < \kappa\},$$

and for all points $o \in \mathbb{H}^2$, we note

$$\mathcal{E}_o(R) := \{v \in T_o^1\mathbb{H}^2 \mid |\zeta_R^v(K) - L(K/\Gamma)| < \kappa\}.$$

A geodesic of length R whose direction is given by a vector $v \in \mathcal{E}(R)$ crosses πK "often", that is at least a number proportional to R. Indeed, if $v \in \mathcal{E}(R)$ we have

$$\frac{1}{R} \text{Leb}\{t \in [0, R] | c_0(t) \cap \pi K \neq \emptyset\} > L(K/\Gamma) - \kappa > \kappa > 0.$$

since $\dot{c}_0(t) \in K$ is equivalent to $c_0(t) \in \pi K$ by definition of K.

The next argument is the key in the proof of Theorem 1.1. It shows that we can compare the length of a geodesic in $\mathbb{H}^2$ which crosses often πK with its d_m length.

Lemma 2.4. *There exists $C > 1$, such that for all $R > 0$, for all $v \in \mathcal{E}_o(R)$ and for all $x \in \{\exp(tv) \mid t \in [R, 2R]\}$, we have :*

$$d_m(o, x) \geq Cd_{\mathbb{H}^2}(o, x). \tag{7}$$

Proof. Let c_0 the geodesic for g_0 and c_m the geodesic for g_m between o and x. Let d be the hyperbolic distance between o and x, $d = d_{\mathbb{H}^2}(o, x)$, and we parametrize c_0 by unit speed we thus have $c_0(d) = x$. Let $N(R)$ be the number of intersections between K and $c_0([0, R])$, that is N is the number of connected component ofs $c_0([0, R]) \cap K$. On one hand, all components of $c_0([0, R]) \cap K$ are inside balls of radius ϵ, hence c_0 "stays" at most 2ϵ in each components. On the other hand, the hypothesis $v \in \mathcal{E}_o(R)$, implies that

$$\frac{1}{R} Leb\{t \in [0, R] | c_0(t) \cap K \neq \emptyset\} > \nu(K) - \kappa = \kappa > 0.$$

These two facts imply that $2\epsilon N(R) \geq \kappa R$, that is to say

$$N(R) \geq \frac{\alpha}{2\epsilon} R. \tag{8}$$

For $i \leq N(R)$, let $t_i \in [0, d]$ such that $c_0(t_i) \in K$ and $c_0[t_{i-1}, t_i] \cap K \neq \emptyset$ is connected. We just have chosen a point $x_i = c_0(t_i)$ in each balls of K crossing c_0. There exists γ_i such that
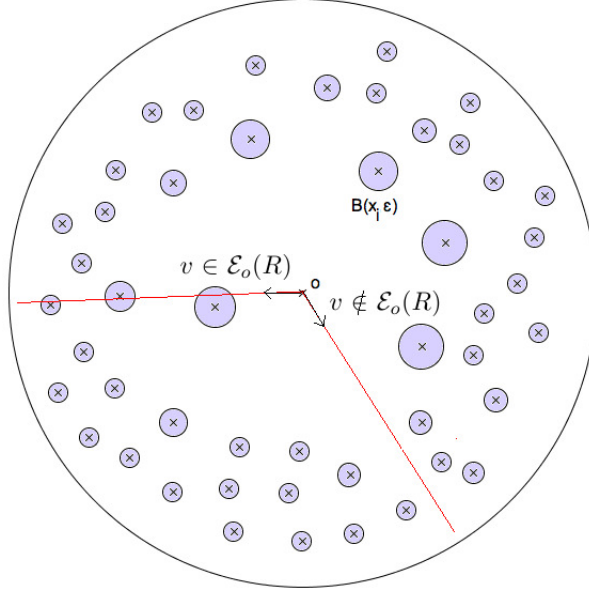
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Figure 1: $\Gamma \cdot B(x_i, \varepsilon)$, $\mathcal{E}_o(R)$ et $\mathcal{E}_o^c(R)$.

$x_i \in B(\gamma_i x, \epsilon)$ hence $B(x_i, \epsilon) \subset B(\gamma_i x, 2\epsilon)$. So without loss of generality, changing ϵ by $\epsilon/2$, we can assume that the center of $\gamma_i B(x, \epsilon)$ lies on c_0.

We have a partition of c_0, by geodesic segments $[x_i, x_{i+1}]$, such that every x_i is the center of a ball of radius $\varepsilon > 0$, on which $g_m = \cosh^2(\eta) g_0$. We call y_i the middle of $[x_i, x_{i+1}]$. We now restraint our attention on one segment $[y_i, y_{i+1}]$. Let $0 < a < 1$ whose dependence on η will be made clear in the rest of the proof. We are going to analyse two different cases.

Assume c_m crosses $B(x_i, a\epsilon)$

Let Δ_i be the lines (geodesics in $\mathbb{H}^2$) orthogonal to c_0 and passing through y_i. Let z_i^1 and z_i^2 be the diameter of $B(x_i, \epsilon)$ defined by $z_i^1 = c_0(t_i - \epsilon)$ and $z_i^2 = c_0(t_i + \epsilon)$, and call D_i^1 and D_i^2 the lines orthogonal to c_0 and passing through z_i^1 and z_i^2. Let A_i, B_i, C_i and A_{i+1} the intersections of c_m and respectively Δ_i, D_i^1 , D_i^2 and Δ_{i+1}. This divides c_m in three connected components, our work will be to give a lower bound for the length of each components cf. Figure 2.

The g_m-length of c_m from A_i to B_i is equal to its g_0-length since the metric g_m is equal to the metric g_0 outside K. Moreover the g_0-length of c_m from A_i to B_i is greater than $d_{g_0}(y_i, z_i^1)$ since the orthogonal projection decreases lengths. We then have

$$d_m(A_i, B_i) \geq d_{g_0}(y_i, z_i^1).$$

For the same reasons we have

$$d_m(C_i, A_{i+1}) \geq d_{g_0}(z_i^2, y_{i+1}).$$

We want to give a lower bound for the g_m-length of c_m between B_i and C_i. We made the assumption that c_m crosses the ball $B(x_i, a\varepsilon)$ hence c_m stays at least $2\varepsilon - 2a\varepsilon$ in the ball $B(x_i, \varepsilon)$. In other words if c_m is unitary *for g_0* we have $\text{Leb}\{t \mid c_m(t) \cap B(x_i, \varepsilon) \neq \emptyset\} \geq 2\varepsilon - 2a\varepsilon$. In the

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ball $B(x_i, \varepsilon)$, the metric g_m is equal to $\cosh(\eta)^2 g_0$ hence the g_m-length satisfies

$$
\begin{aligned}
d_m(B_i, C_i) &\geq \int_{\{t \mid c_m(t) \cap B(x_i, \varepsilon) \neq \emptyset\}} \|\dot{c}_m(t)\|_m dt \\
&= \int_{\{t \mid c_m(t) \cap B(x_i, \varepsilon) \neq \emptyset\}} \cosh(\eta) \\
&\geq \varepsilon \cosh(\eta)(2 - 2a).
\end{aligned}
$$

Choose $a > 0$ such that $\cosh(\eta)(2\varepsilon - 2a\varepsilon) > 2\varepsilon$, that is to say $a \leq 1 - \frac{1}{\cosh(\eta)}$. In order to fix the idea we set $a := \frac{1}{2}(1 - \frac{1}{\cosh(\eta)})$. This implies

$$
\begin{aligned}
d_m(B_i, C_i) &\geq \varepsilon \cosh(\eta)(2 - 2a) \\
&= \varepsilon \cosh(\eta) \left(2 - \left(1 - \frac{1}{\cosh(\eta)} \right) \right) \\
&= (\cosh(\eta) + 1)\varepsilon \\
&= 2\varepsilon + \varepsilon[\cosh(\eta) - 1)] \\
&= d_{g_0}(z_i^1, z_i^2) + \varepsilon[\cosh(\eta) - 1)].
\end{aligned}
$$

Finally we proved

$$
d_m(A_i, A_{i+1}) \geq d_{g_0}(y_i, y_{i+1}) + \varepsilon[\cosh(\eta) - 1]. \tag{9}
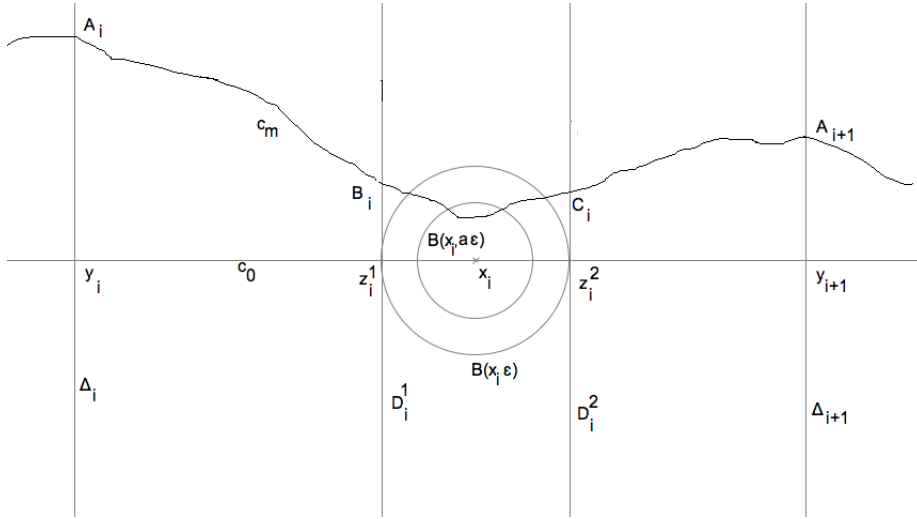$$

Figure 2: c_m crosses $B(x_i, a\varepsilon)$.

Assume c_m does not cross $B(x_i, a\varepsilon)$

Let Δ_i be the line orthogonal to c_0 and passing through y_i and Ω_i the one through x_i. Call A_i the interesection of c_m and Δ_i and E_i the intersecton of c_m with Ω_i. Since c_m does not cross

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$B(x_i a\epsilon)$, E_i is in one of the connected component of $\Omega_i \backslash B(x_i, a\epsilon)$. Named e_i the intersection of $B(x_i, a\epsilon)$ and Ω_i in the same connected component as E_i, this is also the orthogonal projection of W_i on $B(x_i, a\epsilon)$. We parametrise the geodesic Ω_i by $\mathbb{R}$, we give $\omega : \mathbb{R} \to \mathbb{H}^2$ such that $\omega(\mathbb{R}) = \Omega_i$. We suppose that $\omega(0) = x_i$ and the orientation is chosen in order to have $\omega(a\epsilon) = e_i$ The function $t \to d_{g_0}(\omega(t), \Delta_i)$ is convex, which as a minimum at 0, it is hence increasing on $\mathbb{R}^+$. Therefore, $d_{g_0}(\Delta_i, E_i) \geq d_{\mathbb{H}^2}(\Delta_i, e_i)$. It follows that

$$d_m(A_i, E_i) \geq d_{\mathbb{H}^2}(A_i, E_i) \geq d_{g_0}(\Delta_i, E_i) \geq d_{g_0}(\Delta_i, e_i).$$

Let us compute $d_{g_0}(\Delta_i, e_i)$. We fix the following notations :

$$
\begin{aligned}
L &= d_{g_0}(\Delta_i, e_i) \\
l &= d_{g_0}(y_i, x_i) \\
H &= d_{g_0}(y_i, e_i)
\end{aligned}
$$

Now Pythagore's theorem in hyperbolic geometry for the triangle $(y_i x_i e_i)$ gives

$$\cosh(l)\cosh(a\varepsilon) = \cosh(H).$$

Let θ be the angle $\widehat{x_i y_i e_i}$. We have

$$\cos(\theta) = \frac{\tanh(l)}{\tanh(H)},$$

and

$$\sin(\pi/2 - \theta) = \frac{\sinh(L)}{\sinh(H)}.$$

Hence

$$
\begin{aligned}
\sinh(L) &= \sinh(H)\frac{\tanh(l)}{\tanh(H)} \\
&= \cosh(H)\tanh(l) \\
&= \cosh(a\varepsilon)\sinh(l).
\end{aligned}
$$

From this equation, we *cannot* conclude that $L > l + u$ for some $u > 0$. Indeed if L goes to 0 so does l. To avoid this problem we are going to assume that l is greater than the injectivity radius of S.

Remark the following property of sinh which is a consequence of easy calculus. For all $x_0 > 0$ and $\varpi > 1$, there exists $u > 0$, such that for all $x > x_0$, we have $\varpi \sinh(x) \geq \sinh(x + u)$. Now we can choose y_i on c_0 in order to have $d_{g_0}(x_i, y_i) \geq s/2$ where s is the injectivity radius of $\mathbb{H}^2/\Gamma$. Consequently, applying the previous property with $\varpi = \cosh(a\varepsilon)$ and $x_0 = s/2$, there exists $u > 0$ such that.

$$\cosh(a\varepsilon)\sinh(l) \geq \sinh(l + u).$$

Since sinh is increasing we deduce that

$$L \geq l + u.$$

Altogether, we show that there exists $u > 0$ such that

$$d_m(A_i, E_i) \geq d_{g_0}(y_i, x_i) + u.$$

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By the same arguments we can show that

$$d_m(E_i, A_{i+1}) \geq d_{g_0}(x_i, y_{i+1}) + u.$$

Hence, if c_m does not meet $B(x_i, a\varepsilon)$, the g_m-length of c_m between A_i and A_{i+1} satisfies,

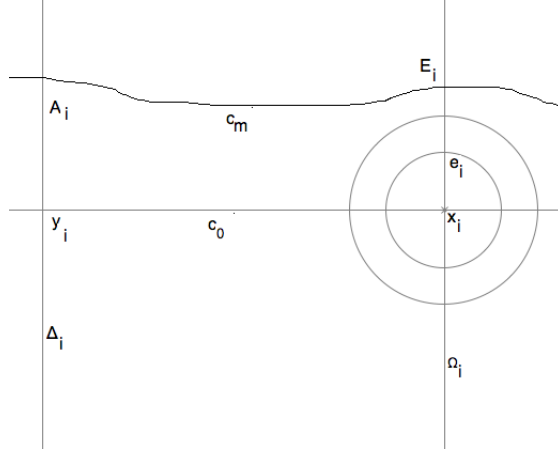$$d_m(A_i, A_{i+1}) \geq d_{g_0}(y_i, y_{i+1}) + 2u. \tag{10}$$

Figure 3: c_m does not cross $B(x_i, a\varepsilon)$.

Conclusion   Let $\alpha := \min\{\varepsilon[\cosh(\eta) - 1]; 2u\}$. From (9) and (10) we have :

$$d_m(A_i, A_{i+1}) \geq d_{g_0}(y_i, y_{i+1}) + \alpha.$$

Summing on i we get

$$d_m(o, x) \geq d_{g_0}(o, x) + N(R)\alpha.$$

Equation (8) and the fact that $d_{g_0}(o, x) \leq 2R$ [1] imply that

$$N(R) \geq \frac{\kappa}{2\varepsilon} R \geq \frac{\kappa}{4\varepsilon} d_{g_0}(o, x).$$

Subsequently,

$$d_m(o, x) \geq \left(1 + \frac{\alpha\kappa}{4\varepsilon}\right) d_{g_0}(o, x).$$

This proves the Lemma with $C = \left(1 + \frac{\alpha\kappa}{4\varepsilon}\right)$. $\qquad\square$

We are now going to compare the entropy of $(\mathbb{D}, d_m)$ with the one of $\mathbb{H}^2$. Let us define

$$\mathcal{F}_o(R) = \{\exp(tv) \,|\, t \in \mathbb{R}^+, v \in \mathcal{E}_o(R)\}.$$

We note by $B_m(o, 2R)$ the ball of radius $2R$ for the d_m distance.

---

[1]this is where we use the upper bound on $d_{g_0}(o, x)$.

Lemma 2.5. *Let $C' := \min(2, C)$ where C satisfies the Lemma 2.4. We have for all $o \in \mathbb{D}$, and all $R > 0$:*

$$B_m(o, 2R) \subset B_{\mathbb{H}^2}(o, 2R/C') \cup \Big(B_{\mathbb{H}^2}(o, 2R) \cap \mathcal{F}_o^c(R)\Big).$$

Proof. Indeed we have $B_m(o, 2R) = \Big(B_m(o, 2R) \cap \mathcal{F}_o(R)\Big) \cup \Big(B_m(o, 2R) \cap \mathcal{F}_o^c(R)\Big)$. Let $x \in B_m(o, 2R) \cap \mathcal{F}_o(R)$. Since $d_{\mathbb{H}^2}(o, x) \leq d_m(o, x)$, it follows that $d_{\mathbb{H}^2}(o, x) \leq 2R$. There are only two possibilities. If $d_{\mathbb{H}^2}(o, x) \leq R$, we have in particular $d_{\mathbb{H}^2}(o, x) \leq \frac{2R}{C'}$. However, if $d_{\mathbb{H}^2}(o, x) \geq R$, we apply Lemma 2.4 and we get $d_{\mathbb{H}^2}(o, x) \leq \frac{2R}{C} \leq \frac{2R}{C'}$. Therefore,

$$B_m(o, 2R) \cap \mathcal{F}_o(R) \subset B_{\mathbb{H}^2}(o, \frac{2R}{C'}) \cap \mathcal{F}_o(R) \subset B_{\mathbb{H}^2}(o, \frac{2R}{C'}).$$

Since we also have for $R > 0$, $B_m(o, 2R) \subset B_{\mathbb{H}^2}(o, 2R)$, this gives

$$B_m(o, 2R) \cap \mathcal{F}_o^c(R) \subset B_{\mathbb{H}^2}(o, 2R) \cap \mathcal{F}_o^c(R),$$

and prove the lemma. $\qquad\square$

The Liouville measure on $T^1\mathbb{H}^2$ is the product of the riemannian measure of $\mathbb{H}^2$ with the angular measure on every fiber. We denote this product by $L = d\mu(x) \times d\theta(x)$. Our aim is to show that the set $\mathcal{E}_o^c(R)$ is small and the volume of $\Big(B_{\mathbb{H}^2}(o, 2R) \cap \mathcal{F}_o^c(R)\Big)$ is small compared to the one of $B_{\mathbb{H}^2}(o, 2R)$. For this we are going to use a large deviation theorem of Y. Kifer [] which gives an upper bound on the mass of the vector which does not behave as the Liouville measure.

Let $\mathcal{P}$ be the set of invariant probabilities on $T^1\mathbb{H}^2/\Gamma$. We also denote by L the Liouville measure on the quotient $T^1\mathbb{H}^2/\Gamma$ and for a vector $v \in T^1\mathbb{H}^2/\Gamma$ we note by ζ_v^R the probability measure given for all borelians subset $E \subset T^1\mathbb{H}^2/\Gamma$ by

$$\zeta_R^v(E) = \frac{1}{R}\int_0^R \chi_E\left(\phi_t^{\mathbb{H}^2/\Gamma}(v)\right) dt.$$

Theorem 2.6. *[Kif90, Théorème 3.4] Let A be a compact subset of $\mathcal{P}$,*

$$\limsup_{T \to \infty} \frac{1}{T}\log L\left\{v \in T^1\mathbb{H}^2/\Gamma \mid \zeta_v^T \in A\right\} \leq -\inf_{\mu \in A \cap \mathcal{P}^t} f(\mu)$$

where $f(\mu) = 1 - h_\mu(\phi_t^{\mathbb{H}^2/\Gamma})$ and $h_\mu(\phi_t^{\mathbb{H}^2/\Gamma})$ is the entropy of the geodesic flow $\phi_t^{\mathbb{H}^2/\Gamma}$ with respect to μ.

Lemma 2.7. *There exists $o \in \mathbb{H}^2$, $\alpha > 0$ and $R_0 > 0$ such that for all $R > R_0$*

$$\theta_o(\mathcal{E}_o^c(R)) \leq e^{-\alpha R}.$$

Proof. Let us keep the notations of Lemma 2.4, $K = T^1\Gamma \cdot B(x, \varepsilon)$ and we consider the following compact subset of $\mathcal{P}$

$$A := \{\mu \in \mathcal{P} \mid |\mu(K/\Gamma) - L(K/\Gamma)| \geq \kappa\}.$$

Since $L \notin A$ and L is the unique measure of maximal entropy satisfying $h(L) = 1$, we have

$$-\inf_{\mu \in A} f(\mu) = -\alpha < 0.$$

Besides, it is clear that the set $\mathcal{E}^c(R) = \{v \in T^1\mathbb{H}^2 \mid |\zeta_R^v(K) - L(K/\Gamma)| \geq \kappa\}$ is Γ-invariant from the Γ invariance of K. By definition we get

$$\mathcal{E}^c(R)/\Gamma = \left\{v \in T^1\mathbb{H}^2/\Gamma \mid \zeta_v^R \in A\right\}.$$

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The Theorem 2.6 says that there exists $R_0 > 0$ such that for all $R > R_0$ we have

$$L(\mathcal{E}^c(R)/\Gamma) \leq e^{-\alpha R}.$$

The product structure of L implies the existence of a point $o \in \mathbb{H}^2/\Gamma$ such that

$$\theta_o\left(\mathcal{E}_o^c(R)/\Gamma\right) \leq e^{-\alpha R}.$$

The Lemma follows, choosing any lift of o in $\mathbb{H}^2$. $\qquad\square$

We finish the proof of Theorem 1.1 with Lemma 2.8, which compare the critical exponent between d_m and hyperbolic distance. Lemmas 2.2 and 2.8, conclude the proof.

Lemma 2.8. *There exists $u > 0$ such that*

$$\delta((\mathbb{D}, d_m)) \leq 1 - u.$$

Proof. We are going to show that the volume entropy of $(\mathbb{D}, d_m)$ satisfies the inequality, that would imply the similar result on critical exponent.

Let $o \in \mathbb{D}$ be a point satisfying Lemma 2.7. From Lemma 2.5, we have

$$B_m(o, 2R) \subset B_{\mathbb{H}^2}(o, \frac{2R}{C'}) \cup \left(B_{\mathbb{H}^2}(o, 2R) \cap \mathcal{F}_o^c(R)\right).$$

On one hand we have the classical upper bound $\mathrm{Vol}\left(B_{\mathbb{H}^2}(o, \frac{2R}{C'})\right) = O(e^{2R/C'})$. On the other hand the volume form on $\mathbb{H}^2$ can be written in polar coordinates as $\sinh(r)drd\theta$, hence for all $R > R_0$ we get

$$
\begin{aligned}
\mathrm{Vol}\left(B_{\mathbb{H}^2}(o, 2R) \cap \mathcal{F}_o^c(R)\right) &= \int_0^{2R} \int_{\mathcal{E}_o^c(R)} \sinh(r)d\theta dr \\
&\leq \int_0^{2R} e^{-\alpha R} e^r dr \\
&\leq e^{(2-\alpha)R}.
\end{aligned}
$$

Let $u > 0$, defined by $1 - u = \max(\frac{1}{C'}, (1 - \alpha/2)) < 1$. The last two upper bounds give

$$
\begin{aligned}
\mathrm{Vol}(B_m(o, 2R)) &= O(e^{2R/C'}) + O(e^{(2-\alpha)R}) \\
&= O(e^{2(1-u)R})
\end{aligned}
$$

We finish by taking the log and the limit. $\qquad\square$

3 Quasi-Fuchsian case

3.1 Geodesic intersection

Let Σ be an incompressible surface in M. We designed by $\phi_t^{\mathbb{H}^3}$, ϕ_t^{Σ} the geodesic flows on the unitary tangent spaces $T^1\mathbb{H}^3$, $T^1\Sigma$ respectively. We named π the projection from $T^1\mathbb{H}^3$ to $\mathbb{H}^3$. The restriction of π to $T^1\Sigma$ will still be denoted by π. There is two distances we can consider on Σ. The intrinsic one, defined as the infimum of the length of curves staying on Σ and the extrinsic one, where we take the distance in $\mathbb{H}^3$. We will denote d_Σ and d this two distances.

First of all let us remark that there is no riemanniann metric on Σ which induces d. If such a metric existed, the 1.3 would be a particular case of [Kni95]

Proposition 3.1. *If Σ is not totally geodesic, there is no riemannian metric on Σ which induces d.*

Proof. Assume there is such a riemannian metric, named g'. Let $c_{g'} : [0,1] \to \Sigma$ a geodesic for g' on Σ, then for all $x \in [0,1]$,

$$d_{g'}(c_{g'}(0), c_{g'}(x)) + d_{g'}(c_{g'}(x), c_{g'}(1)) = d_{g'}(c_{g'}(0), c_{g'}(1))$$

But since we suppose that g' induces d we have the same equality for d

$$d(c_{g'}(0), c_{g'}(x)) + d(c_{g'}(x), c_{g'}(1)) = d(c_{g'}(0), c_{g'}(1))$$

and this implies that $c_{g'}$ is a geodesic for $\mathbb{H}^3$. Hence all geodesics of Σ are geodesics of $\mathbb{H}^3$ that is to say, Σ is totally geodesic. $\qquad\square$

Consider the following function

$$a : \left| \begin{array}{ccc} T^1\Sigma \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ (v,t) & \longmapsto & d(\pi\phi_t^\Sigma(v), \pi(v)) \end{array} \right.$$

Let $t_1, t_2 \in \mathbb{R}$ and $v \in T^1\Sigma$, we have by triangular inequality,

$$\begin{aligned} a(v, t_1 + t_2) &= d(\pi\phi_{t_1+t_2}^\Sigma(v), \pi(v)) \\ &\leq d(\pi\phi_{t_1+t_2}^\Sigma(v), \pi\phi_{t_1}^\Sigma(v)) + d(\pi\phi_{t_1}^\Sigma(v), \pi(v)) \\ &\leq d(\pi\phi_{t_2}^\Sigma(\phi_{t_1}v), \pi\phi_{t_1}^\Sigma(v)) + d(\pi\phi_{t_1}^\Sigma(v), \pi(v)) \\ &\leq a(\phi_{t_1}^\Sigma v, t_2) + a(v, t_1) \end{aligned}$$

hence a is a subadditive cocycle for the geodesic flow ϕ_t^Σ. Since a is Γ invariant it defines a subbadittive cocycle on $T^1\Sigma$, still denoted by a.

The following is a consequence of the subadditive ergodic theorem []

Theorem 3.2. *Les μ be a ϕ_t^Σ invariant probability measure on $T^1\Sigma$. Then*

$$I_\mu(\Sigma, M, v) := \lim_{t\to\infty} \frac{a(v,t)}{t}$$

exists for μ almost $v \in T^1\Sigma$ and defines a μ-integrable function on $T^1\Sigma$, invariant under the geodesic flow and we have :

$$\int_{T^1\Sigma} I_\mu(\Sigma, M, v) d\mu = \lim_{t\to\infty} \int_{T^1\Sigma} \frac{a(v,t)}{t} d\mu.$$

Moreover if μ is ergodic $I_\mu(\Sigma, M, v)$ is constant μ-almost everywhere. In this case, we write $I_\mu(\Sigma, M)$

3.2 Patterson Sullivan measures

We called Λ the limit set of Γ acting on $\mathbb{H}^3$. Since Γ acts cocompactly on Σ, and on the convex core $C(\Lambda)$, the three geometric spaces Γ, Σ and $C(\Lambda)$ are quasi-isometric. We assume from now on that (Σ, g) has negative curvature, hence there is a unique geodesic in each homotopy class of curves, and for every pair of points in Σ there is a unique geodesic which joints them. Let c_Σ be a geodesic on Σ, and denoted by $c_\Sigma(\pm\infty)$ its limit points on Λ. There is a unique $\mathbb{H}^3$-geodesic

$c_{\mathbb{H}^3}$ whose endpoints are $c_\Sigma(\pm\infty)$. Since Σ is quasi-isometric to $C(\Lambda)$, the two geodesics $c_{\mathbb{H}^3}$ and c_Σ are at bounded distance.

Let $p \in \Sigma$ and call pr_p^Σ the projection from Σ to Λ defined as follows. For any point $x \in \Sigma$ call $c_{p,x}^\Sigma$ the geodesic on Σ which joint p to x, then

$$pr_p^\Sigma(x) = c_{p,c}^\Sigma(+\infty).$$

We will denote the equivalent projection in $\mathbb{H}^3$ by $pr_p^{\mathbb{H}^3}$. There is two small distinctions to notice between $pr_p^{\mathbb{H}^3}$ and pr_p^Σ. First $pr_p^{\mathbb{H}^3}$ is defined for every points in $\mathbb{H}^3$, whereas pr_p^Σ is only defined for points in Σ. Second is that the arrival set of pr_p^Σ is exactly Λ whereas the arrival set of $pr_p^{\mathbb{H}^3}$ is all S^2.

As we just said, for all $\xi \in \Lambda$ the geodesics, $c_{p,\xi}^\Sigma$ and $c_{p,\xi}^{\mathbb{H}^3}$ are at bounded distance, and this bound depends only on the quasi-isometry between Σ and $C(\Lambda)$. There exists C_1 such that for all $\xi \in \Lambda$ the Hausdorff distance between geodesics $c_{p,\xi}^\Sigma$ and $c_{p,\xi}^{\mathbb{H}^3}$ is less than C_1. Let $x \in \Sigma$, $R > 0$ and consider the ball $B_{\mathbb{H}^3}(x, R)$ in $\mathbb{H}^3$ of center x and radius R. Now take $\xi \in pr_p^{\mathbb{H}^3}(B(x, R-C_1)) \cap \Lambda$, this means that the $\mathbb{H}^3$-geodesic from p to ξ cross the ball $B_{\mathbb{H}^3}(x, R - C_1)$. This $\mathbb{H}^3$-geodesic is at bounded distance C_1 of the Σ-geodesic joining p to ξ. Hence $c_{p,\xi}^\Sigma \cap (B_{\mathbb{H}^3}(x, R) \cap \Sigma) \neq \emptyset$, this proves that $\xi \in pr_p^\Sigma(B_{\mathbb{H}^3}(x, R) \cap \Sigma)$.

The same argument shows that

$$pr_p^\Sigma(B_{\mathbb{H}^3}(x, R) \cap \Sigma) \subset pr_p^{\mathbb{H}^3}(B_{\mathbb{H}^3}(x, R+C_1)) \cap \Lambda \subset pr_p^{\mathbb{H}^3}(B_{\mathbb{H}^3}(x, R+C_1)).$$

The distance on Σ and on $\mathbb{H}^3$ are locally equivalent, for every $R > 0$ there exists C_2 such that all balls satisfy the following

$$B_\Sigma(x, R - C_2) \subset B_{\mathbb{H}^3}(x, R) \cap \Sigma \subset B_\Sigma(x, R + C_2)$$

Set $C = \max(C_1, C_2)$ we then have

Theorem 3.3.

$$pr_p^\Sigma(B_\Sigma(x, R - C))$$
$$\cap$$
$$pr_p^{\mathbb{H}^3}(B_{\mathbb{H}^3}(x, R-C)) \cap \Lambda \quad \subset \quad pr_p^\Sigma(B_{\mathbb{H}^3}(x, R) \cap \Sigma) \quad \subset \quad pr_p^{\mathbb{H}^3}(B_{\mathbb{H}^3}(x, R+C))$$
$$\cap$$
$$pr_p^\Sigma(B_\Sigma(x, R + C))$$

Before proving Theorem 1.3, we will recall some basic facts about Patterson-Sullivan measure. Let (X, g) be a simply connected manifolds with negative curvature and $X(\infty)$ its geometric boundary. If Γ is a discrete group acting on (X, g) we can associated a family of measures $\{\mu_p^g\}_{p \in X}$ on $X(\infty)$ constructed as follows. Let x, y two points of X and consider the Poincaré series :

$$P(s) := \sum_{\gamma \in \Gamma} e^{-sd(\gamma x, y)}.$$

The convergence of $P(s)$ is independent of x and y by the triangular inequality. It converges for $s > \delta(\Gamma)$ and diverges for $s < \delta(\Gamma)$. If the action is cocompact, $\delta(\Gamma) = h(g)$ and the series diverges at $h(g)$. Then we define the probability measure

$$\mu_{p,x}^g(s) := \frac{\sum_{\gamma \in \Gamma} e^{-sd(\gamma x, p)} \delta_{\gamma x}}{\sum_{\gamma \in \Gamma} e^{-sd(\gamma p, p)}}.$$

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We obtain a measure on $X(\infty)$ when we consider the weak limit [2]

$$\mu_p^g := \lim_{s_n \to h(g)} \mu_p^g(s_n)$$

It is concentrated on the accumulation points of G, that is to say the limit set.

These measures called *Patterson-Sullivan measures* have the following properties. They are quasi-conformal,

$$\frac{d\mu_p^g}{d\mu_q^g}(\xi) = e^{-h(g)\beta_\xi(p,q)},$$

where $\beta_\xi(p,q) = \lim_{z \to \xi} d_g(p,z) - d_g(q,z)$.

And Γ-equivriant,

$$\mu_p^g \circ \gamma = \mu_{\gamma^{-1}p}^g$$

for all $\gamma \in \Gamma$.

Moreover we know how these measures behave locally,

Lemma 3.4 (Shadow's lemma). *For every R there exists $c > 1$ such that for all $x \in X$*

$$\frac{1}{c}e^{-h(g)d_g(x,p)} \le \mu_p^g(pr_p^g(B_g(x,R))) \le ce^{-h(g)d_g(x,p)}.$$

Suppose that X/Γ is compact, from Patterson-Sullivan measure, we can construct a invariant probability on T^1X/G. Let $\Lambda^{(2)}$ be $\Lambda \times \Lambda - diagonal$, there is a natural identification of $\Lambda^{(2)} \times \mathbb{R}$ and T^1X, a vector $v \in T^1X$ is identified with $(c_v(+\infty), c_v(-\infty), \beta_{c_v(+\infty)}(p, \pi v))$. The Bowen-Margulis measure is defined by

$$d\mu_{BM}(\xi, \eta, t) = e^{2h(g)\langle \xi|\eta \rangle_p} d\mu_p^g(\xi) d\mu_p^g(\eta) dt$$

where $\langle \xi|\eta \rangle_p$ is the Gromov product, given by

$$\langle \xi|\eta \rangle_p = \lim_{t \to \infty}(t - \frac{1}{2}d(c_{p,\xi}(t), c_{p,\eta}(t)).$$

From the quasi-conformal property of μ_p^g, it follows that μ_{BM} is invariant by Γ and defined a measure on T^1X/Γ. The invariance by the geodesic flow is clear by definition and it is shown in [Nic89] that μ_{BM} is ergodic.

Finally we will need the following theorem.

Theorem 3.5. *If μ_p^Σ and $\mu_p^{\mathbb{H}^3}$ are equivalent, then the marked length spectrum of Σ is proportional to the marked length spectrum of M.*

Before proving this theorem let us introduce Gromov distance on the boundary.

Definition 3.6. *[GdlH90] Let ξ, η be two points on ∂X. The Gromov distance is defined by*

$$D_X(\xi, \eta) = \exp(-a \langle \xi|\eta \rangle_p),$$

where $a > 0$ is sufficiently small for D_X to be distance.

---

[2]It is a classical result of Sullivan that there is in fact a unique limit, this follows from the ergodicity of Bowen Margulis measure.

Remark that for the same space, there is not a unique Gromov distance, but two of them are Hölder equivalent.

The proof of Theorem 3.5 is in two steps. The first one we prove that if the Patterson Sullivan measure are equivalent then the Gromov distances D_Σ and $D_{\mathbb{H}^3}$ are Hölder equivalent. The second one we prove that this last condition implies the proportionality of the length spectrum.

Lemma 3.7. *If μ_p^Σ and $\mu_p^{\mathbb{H}^3}$ are equivalent, then the two Gromov distances are Hölder equivalent.*

Proof. Let us consider on $\Lambda^{(2)}$ the Bowen-Margulis currents defined by

$$\nu_\Sigma(\xi, \eta) = \frac{d\mu_\Sigma^p(\xi) d\mu_\Sigma^p(\eta)}{D_\Sigma(\xi, \eta)^{2\delta(\Sigma)}}$$

$$\nu_{\mathbb{H}^3}(\xi, \eta) = \frac{d\mu_{\mathbb{H}^3}^p(\xi) d\mu_{\mathbb{H}^3}^p(\eta)}{D_{\mathbb{H}^3}(\xi, \eta)^{2\delta(\mathbb{H}^3)}}.$$

From the definitions of Busemann functions ($\beta_\xi(x) = \lim_{u \to \xi} d(x, u) - d(o, u)$) and Gromov product ($< \xi, \eta > = \lim_{u \to \xi, v \to \eta} \frac{1}{2}(d(u, o) + d(v, o) - d(u, v))$) we have for Σ as well as for $\mathbb{H}^3$ that

$$\beta_\xi(\gamma p) + \beta_\eta(\gamma p) = 2 < \gamma^{-1}\xi, \gamma^{-1}\eta > -2 < \xi, \eta > .$$

Using $D(\xi, \eta) = e^{-<\xi, \eta>}$ and conformal property of Patterson Sullivan measures, the currents ν_Σ and $\nu_{\mathbb{H}^3}$ are Γ-invariant.

By assumption μ_Σ^p and $\mu_p^{\mathbb{H}^3}$ are equivalent, therefore ν_Σ and $\nu_{\mathbb{H}^3}$ are also equivalent. The ergodicity and the Γ-invariance implies the existence of $c > 0$ such that

$$\nu_\Sigma = c\nu_{\mathbb{H}^3}.$$

Since μ_p^Σ and $\mu_p^{\mathbb{H}^3}$ are equivalent there exists a function $f : \Lambda \to \mathbb{R}^+$ such that $\mu_p^\Sigma(\xi) = f(\xi)\mu_p^{\mathbb{H}^3}$. We have

$$f(\xi)f(\eta)D_{\mathbb{H}^3}^{\delta(\mathbb{H}^3)}(\xi, \eta) = cD_\Sigma^{\delta(\Sigma)}(\xi, \eta).$$

We see that f is equal almost everywhere to a continuous function. We can therefore suppose that f is continuous on Λ hence strictly positive. By compacity, there exists $C > 1$ such that $\frac{1}{C} \leq f(\xi) \leq C$. Finally we get what we stated

$$\frac{c}{C^2} D_\Sigma^{\delta(\Sigma)}(\xi, \eta) \leq D_{\mathbb{H}^3}^{\delta(\mathbb{H}^3)}(\xi, \eta) \leq C^2 c D_\Sigma^{\delta(\Sigma)}(\xi, \eta).$$

$\square$

We now show the second part

Lemma 3.8. *If D_Σ and $D_{\mathbb{H}^3}$ are Hölder equivalent the marked length spectrum are equivalent*

Proof. In [Cin92], J. P. Otal shows that for all $g \in \Gamma$, and for all $\xi \neq g^\pm \in \Lambda$ we have :

$$[g^-, g^+, g(\xi), \xi] = e^{\ell(g)},$$

where $\ell(g)$ is the displacement of g and $[g^-, g^+, g(\xi), \xi] = \frac{D(g^-, g(\xi))D(g^+, \xi)}{D(g^-, \xi)D(g^+, g(\xi))}$. Applying this result to Σ and $\mathbb{H}^3$ we get

$$[g^-, g^+, g(\xi), \xi]_\Sigma = e^{\ell_\Sigma(g)}$$

and
$$[g^-, g^+, g(\xi), \xi]_{\mathbb{H}^3} = e^{\ell_{\mathbb{H}^3}(g)}.$$

By assumption on the distances $D_\Sigma, D_{\mathbb{H}^3}$, there exists $C > 1$ such that for all $g \in \Gamma$ we have

$$\frac{1}{C} e^{r\ell_{\mathbb{H}^3}(g)} \le e^{\ell_\Sigma(g)} \le C e^{r\ell_{\mathbb{H}^3}(g)}.$$

In particular when we look at the power g^n of g, taking the log, we get for all $n > 0$ and all $g \in \Gamma$:

$$-\log(C) + rn\ell_{\mathbb{H}^3}(g) \le n\ell_\Sigma(g) \le \log(C) + rn\ell_{\mathbb{H}^3}(g).$$

We finish the proof by dividing by n and taking the limit :

$$\ell_\Sigma(g) = r\ell_{\mathbb{H}^3}(g).$$

$\square$

Theorem 3.5 follows directly from Lemmas 3.7 and 3.8.

3.3 Entropy comparison

We finally get to the proof of Theorem 1.3. First we prove the inequality using the behaviour of Patterson-Sullivan measures and a volume comparison of a subset of Σ, the proof follows the same lines as [Kni95, Theorem 3.4]. Then we prove the equality case using Theorem 3.5.

Theorem 3.9. *Let (Σ, g) has negative curvature,*

$$h(g) \le I_{\mu_{BM}}(\Sigma, \mathbb{H}^3)\delta(\Gamma).$$

With equality if and only if the marked length spectrum of Σ is proportional to the marked length spectrum of M.

Proof. The geodesic flow is ergodic with respect to the Bowen-Margulis measure μ_{BM}, hence for μ_{BM}-almost all $v \in T^1\Sigma$ we have :

$$\lim_{t \to \infty} \frac{a(v,t)}{t} = I_\mu(\Sigma, M).$$

Let v and v' be two unit vectors on the same weak stable manifolds. Then $d(c_{v'}(t), c_{v'}(0)) \le d(c_{v'}(t), (c_v(t)) + d(c_v(t), (c_v(0)) + d(c_v(0), (c_{v'}(0))$, and the same inequality holds interchanging the role of v and v'. Moreover $d(c_{v'}(t), (c_v(t))$ decreases exponentially since v and v' are on the same weak stable manifolds. Hence $\lim_{t \to \infty} \frac{a(v,t)}{t}$ exist if and only if $\lim_{t \to \infty} \frac{a(v',t)}{t}$.

Let $v_p(\xi)$ denotes the unitary vector in $T_p^1\Sigma$ such that $c_{v_p(\xi)}(\infty) = \xi$. The previous fact and the product structure of $d\mu_{BM}$ assures that for μ_p^g almost all $\xi \in \partial\Sigma$ we have

$$\lim_{t \to \infty} \frac{a(v_p(\xi),t)}{t} = I_\mu(\Sigma, M).$$

For all $\epsilon > 0$ and $T > 0$ we define the set

$$A_p^{T,\epsilon} = \left\{ \xi \in \partial\Sigma \Big| \left| \frac{a(v_p(\xi),t)}{t} - I_\mu(\Sigma, M) \right| \le \epsilon, \quad t \ge T \right\}.$$

For all $d \in]0, 1[$ and all $\epsilon > 0$, there exists $T > 0$ such that $\mu_p^\Sigma(A_p^{T,\epsilon}) \ge d$. For $t > T$ consider the subset $\{c_{p,\xi}(t) | \xi \in A_p^{T,\epsilon}\} \subset S_g(p,t)$ of the geodesic sphere of radius t and center p on Σ.

Choose $\{B_\Sigma(x_i, R) | i \in I\}$ a covering of this subset of fixed radius $R > 0$ such that $x_i \in S_\Sigma(p, t)$ and $B_\Sigma(x_i, R/4)$ are pairwise disjoint. Then, by the local behaviour of μ_p^Σ, there exists a constant $c > 1$ independent of t such that

$$\frac{1}{c} e^{-h(g)t} \le \mu_p^\Sigma(pr_p^\Sigma(B_\Sigma(x_i, R))) \le c e^{-h(g)t}.$$

It is clear that $A_p^{T,\epsilon} \subset \cup_{i \in I} pr_p^\Sigma(B_\Sigma(x_i, R))$ and therefore,

$$d \le \mu_p^\Sigma \left(\cup_{i \in I} pr_p^\Sigma(B_\Sigma(x_i, R))\right) \le \sum_{i \in I} \mu_p^\Sigma(pr_p^\Sigma(B_\Sigma(x_i, R))) \le c \operatorname{Card}(I) e^{-h(g)t}.$$

Since $\mathbb{H}^3/\Gamma$ is convex cocompact, $C_Q(\Lambda)/\Gamma$ is compact, where $C_Q(\Lambda)$ is the Q neighbourhood of the convex core of Λ. Hence for any $Q > 0$,

$$\delta(\Gamma) = \lim_{R \to \infty} \operatorname{Vol}(B_{\mathbb{H}^3}(o, R) \cap C_Q(\Lambda)).$$

Now take Q sufficiently large such that Σ is inside $C_Q(\Lambda)$. There exists K such that $B_\Sigma(x_i, R/4) \subset B_{\mathbb{H}^3}(x_i, R + K) \cap C_Q(\Lambda)$.

From the definition of the set $A_p^{T,\epsilon}$, we then have that the disjoint union $\cup_{i \in I} B_\Sigma(x_i, R/4) \subset B_{\mathbb{H}^3}(p, t(I_{\mu_{BM}}(\Sigma, \mathbb{H}^3) + \epsilon) + R + K) \cap C_Q(\Lambda)$. It follows that,

$$
\begin{aligned}
e^{h(g)t} &\le \frac{c}{d} \operatorname{Card}(I) \\
&\le \frac{c}{dV} \sum_{i \in I} vol_{\mathbb{H}^3}(B_{\mathbb{H}^3}(x_i, R/4)) \cap C_Q(\Lambda)) \\
&\le \frac{c}{dV} vol_{\mathbb{H}^3}(B_{\mathbb{H}^3}(p, t(I_{\mu_{BM}}(\Sigma, \mathbb{H}^3) + \epsilon) + R + K) \cap C_Q(\Lambda))
\end{aligned}
$$

Hence
$$h(g) \le \frac{1}{t} \left(\log \frac{c}{dV} + \log B_{\mathbb{H}^3}(p, t(I_{\mu_{BM}}(\Sigma, \mathbb{H}^3) + \epsilon) + R + K) \cap C_Q(\lambda)) \right)$$
Taking the limit $t \to \infty$ we get

$$h(g) \le (I_{\mu_{BM}}(\Sigma, \mathbb{H}^3) + \epsilon)\delta(\Gamma)$$

and we conclude since ϵ is arbitrary. $\qquad\square$

For the proof of the equality case in Theorem 1.3 we will use the same lemma as in [Kni95]

Lemma 3.10. *[Kni95] Let $p \in \Sigma$ and μ_p^g the Patterson-Sullivan measure with respect to p and g, there exists a constant L such that for μ_p^g almost all $\xi \in \partial\Sigma$ there is a seqence $t_n \to \infty$ such that*
$$|d(p, \pi\phi_{t_n}^\Sigma v_p(\xi)) - I_{\mu_{BM}}(\Sigma, \mathbb{H}^3)t_n| \le L.$$

Proof. It is shown in G. Knieper, that this property is true if there exists a constant $C > 0$ such that
$$a(v, t_1) + a(\phi_{t_1}^\Sigma v, t_2) \le C + a(v, t_1 + t_2).$$

Let $v \in T^1\Sigma$ and c_v^Σ the geodesic on Σ directed by v. Recall that there exists C_1 such that the $\mathbb{H}^3$-geodesic from $\pi(v)$ to $c_v^\Sigma(t_1 + t_2)$ is at bounded distance C_1 of $c_v^\Sigma(t_1 + t_2)$, independant of t_1 and t_2. The $\mathbb{H}^3$-geodesic from p to $c_v^\Sigma(t_1)$ and the one from $c_v^\Sigma(t_1)$ to $c_v^\Sigma(t_1 + t_2)$ are also at bounded distance C_1 of c_v^Σ. This implies the desired property with $C = 2C_1$. $\qquad\square$

Equality case in 1.3. Suppose that $h(g) = I_{\mu_{BM}}(\Sigma, \mathbb{H}^3)\delta(\Gamma)$. Choose a point $p \in \Sigma$ and $\xi \in \Lambda$, set $y_n := \pi\phi_{t_n}^{\Sigma} v_p(\xi)$. From the above lemma, for μ_p^{Σ} almost all ξ we have

$$|d(p, y_n) - I_{\mu_{BM}}(\Sigma, \mathbb{H}^3)t_n| \leq L.$$

Set $R > 0$ a fixed constant, by local property of the Patterson-Sullivan measure on $\mathbb{H}^3$, there is c_1 such that

$$\frac{1}{c_1}e^{-\delta(\Gamma)d(p,y_n)} \leq \mu_p^{\mathbb{H}^3}(pr_{\mathbb{H}^3}B_{\mathbb{H}^3}(y_n, R)) \leq c_1 e^{-\delta(\Gamma)d(p,y_n)},$$

by Theorem 3.3

$$pr_{\mathbb{H}^3}(B_{\mathbb{H}^3}(x, R-C)) \cap \Lambda \subset pr_{\Sigma}(B_{\mathbb{H}^3}(x, R) \cap \Sigma) \subset pr_{\mathbb{H}^3}(B_{\mathbb{H}^3}(x, R+C)).$$

Hence there is a constant c_2 such that

$$\frac{1}{c_2}e^{-\delta(\Gamma)d(p,y_n)} \leq \mu_p^{\mathbb{H}^3}(pr_{\Sigma}B_{\mathbb{H}^3}(y_n, R) \cap \Sigma) \leq c_1 e^{-\delta(\Gamma)d(p,y_n)}.$$

By the local property of the Patterson-Sullivan measure on Σ, there is c_3 such that

$$\frac{1}{c_3}e^{-h(\Sigma)d_{\Sigma}(p,y_n)} \leq \mu_p^{\Sigma}(pr_{\Sigma}B_{\Sigma}(y_n, R)) \leq c_3 e^{-h(\Sigma)d_{\Sigma}(p,y_n)},$$

and by Theorem 3.3

$$pr_{\Sigma}(B_{\Sigma}(x, R-C)) \subset pr_{\Sigma}(B_{\mathbb{H}^3}(x, R) \cap \Sigma) \subset pr_{\Sigma}(B_{\Sigma}(x, R+C)).$$

Hence there is c_4 such that

$$\frac{1}{c_4}e^{-h(\Sigma)d_{\Sigma}(p,y_n)} \leq \mu_p^{\Sigma}(pr_{\Sigma}B_{\mathbb{H}^3}(y_n, R) \cap \Sigma) \leq c_4 e^{-h(\Sigma)d_{\Sigma}(p,y_n)}.$$

From the choice of y_n and since $h(\Sigma) = I_{\mu_{BM}}(\Sigma, \mathbb{H}^3)\delta(\Gamma)$

$$e^{-L}e^{-\delta(\Gamma)d(p,y_n)} \leq e^{-h(g)d_{\Sigma}(p,y_n)} \leq e^{L}e^{-\delta(\Gamma)d(p,y_n)}.$$

Hence there is $c_5 > 0$ such that

$$\frac{1}{c_5}e^{-\delta(\Gamma)d(p,y_n)} \leq \mu_p^{\Sigma}(pr_{\Sigma}B_{\mathbb{H}^3}(y_n, R) \cap \Sigma) \leq c_5 e^{-\delta(\Gamma)d(p,y_n)}.$$

Finally we have a constant c_6 such that

$$c_6 \leq \frac{\mu_p^{\Sigma}(pr_{\Sigma}B_{\mathbb{H}^3}(y_n, R) \cap \Sigma)}{\mu_p^{\mathbb{H}^3}(pr_{\Sigma}B_{\mathbb{H}^3}(y_n, R) \cap \Sigma)} \leq c_6.$$

Since $pr_{\Sigma}(B_{\mathbb{H}^3}(y_n, R) \cap \Sigma) \to \xi$ the measure μ_p^{Σ} and $\mu_p^{\mathbb{H}^3}$ are equivalent. We conclude by Theorem 3.5. $\square$

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