

CRITICAL EXPONENT OF GRAPHED TEICHMÜLLER REPRESENTATIONS ON $\mathbb{H}^2 \times \mathbb{H}^2$

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ABSTRACT. In this note we survey different results on critical exponent. After giving the general setting and classical known results we study critical exponent associated to a pair of Teichmüller representations acting on $\mathbb{H}^2 \times \mathbb{H}^2$ by diagonal action. We will give new examples of behaviour of this critical exponent. We finally explain the link of this invariant with Anti-De Sitter geometry.

1. INTRODUCTION

Let G be a countable group acting on a metric space X . We are interested in, the *critical exponent of G acting on X* which is defined as the exponential growth rate of an orbit in X . For this notion to be interesting, one has to make some hypothesis on the group, the action and the space. For example the action of G on X should be discrete, and the space X should have a kind of exponential growth property (like the volume of the balls if X is a simply connected Riemannian manifolds). The typical setting where this invariant has been widely studied is the fundamental group of a hyperbolic manifold acting on its universal cover, identified with the hyperbolic space. This is linked to many other invariants as the topological entropy of the geodesic flow, Hausdorff dimension of the limit set, or to the bottom of the spectrum of the Laplacian on X/Γ .

In section 2, we will define the *critical exponent*. We will survey some known results in different settings and explain what are the classical questions associated to this invariant. At the end of this introduction we will describe the one we are interested in : a surface group acting on a product of hyperbolic spaces.

In section 3, we review various examples of the behaviour of this critical exponent $\delta(\cdot, \cdot)$ in the product of Teichmüller spaces. Proofs of most of them can be found in [Glo15]. We present two new results, one concerns the behaviour of δ along a Teichmüller ray, the other concerns its random behaviour when travelling along random walks on Teichmüller space. As a bypass we explain how a Theorem of A. Karlsson on random behaviour of the length of geodesics, [Kar14], can be generalized without technical difficulties.

Section 4 is devoted to the main result in [Glo15], which is an isolation theorem for δ . We give insights on the proof, based on the previous examples.

Finally, section 5 is widely independent of the other and concerns Anti-De Sitter geometry. We will not enter into the details of Lorentzian geometry but try to explain how our results can be applied to the Anti-De Sitter world and are in some sense equivalent to previous known results for hyperbolic manifolds.

Sections 2 and 5 are intended for non specialists. Section 3 and 4 are more demanding and a background in Teichmüller geometry via geodesic currents (cf. [Bon88]) would be useful, however we try to explain every used objects.

Except for new results, we will not give complete proofs, our aim is to give ideas on how the proofs work. The interested reader can find them in the given references.

2. CRITICAL EXPONENT

The aim of this section is to recall definitions and classical results on critical exponent. A good reference for this notion are the articles of M. Peigné, T. Roblin and S. Tapie, and F. Paulin [Pei10, RT13, Pau13] (all in French) and the text of K. Matsuzaki [Mat01] (in English). We finish by defining the critical exponent we are looking at in the rest of the paper.

2.1. Definitions. Let (X, d) be a locally compact, metric space. The reader should think to the usual setting when X is a simply connected Riemannian manifold of non positive curvature. However we will consider in section 3 and 4, $\mathbb{H}^2 \times \mathbb{H}^2$ endowed with the distance defined by the sum of the distance on each factor, instead of the Riemannian product. Therefore we give a slightly general definition.

Definition 2.1. Let o be a point in (X, d) and Γ a countable subgroup of isometries of X . We call Poincaré series the function :

$$P(s) := \sum_{\gamma \in \Gamma} e^{-sd(\gamma o, o)}.$$

The critical exponent is the number $\delta \geq 0$ defined by

$$\delta(\Gamma \curvearrowright X) := \inf\{s > 0 \mid P(s) < +\infty\}.$$

The critical exponent δ does not depend on the base point thanks to the triangular inequality. Moreover, when there will be no doubt on the group Γ (resp. the space X , resp both) we are considering, we will write $\delta(X)$ (resp. $\delta(\Gamma)$, resp. δ) instead of $\delta(\Gamma \curvearrowright X)$.

A simple computation on power series [Pei10] shows that the critical exponent corresponds to the exponential growth rate of points of an orbit, that lies in a ball of radius R :

$$(1) \quad \delta = \limsup_{R \rightarrow \infty} \frac{1}{R} \log \text{Card } \Gamma.o \cap B_X(o, R).$$

We can deduce from this fact that for cocompact action the critical exponent is equal to the volume entropy $h(X)$ defined as the exponential growth rate of the volume of balls :

$$h(X) := \limsup_{R \rightarrow \infty} \frac{1}{R} \log \text{Vol } B_X(o, R).$$

Proposition 2.2. [Pei10] Let (X, g) be a simply connected Riemannian manifold. Let $\Gamma < \text{Isom}(X)$ be a discrete group, such that X/G is compact. Then

$$\delta(\Gamma \curvearrowright X) = h(X).$$

The proof consists in covering $B_X(o, R)$ by translates of a fundamental domain and then compare the volume of the balls and the union of the translates. It cannot be generalized to non-uniform lattice ; when X/G is not compact but of finite volume where $\delta(\Gamma \curvearrowright X) < h(X)$ might happened. A counter-example can be found in [Pei10].

We give a first example where it is easy to compute the critical exponent thanks to the previous proposition. Let S be closed surface of genus $g \geq 2$ and denote by $\Gamma := \pi_1(S)$ its fundamental group. By the uniformisation theorem, S admits a hyperbolic metric g or, in other words, there exists a faithful and discrete representation ρ of Γ into $\text{Isom}^+(\mathbb{H}^2)$ such that $(S, g) \simeq \mathbb{H}^2/\rho(\Gamma)$. We then have

$$\delta(\Gamma \curvearrowright_{\rho} \mathbb{H}^2) = h(\mathbb{H}^2) = 1,$$

since we have $B_{\mathbb{H}^2}(o, R) := 2\pi \cosh(R)$.

More generally, if M is a compact hyperbolic manifolds of dimension n and $\Gamma := \pi_1(M)$ acts on \mathbb{H}^n then $\delta = n - 1$.

2.2. Classical results. There are essentially two ways to generalize this result. We can fix \mathbb{H}^n and play with smaller groups, or fix the group and play with other spaces. The first question has been intensively studied, let us cite at least [Pat76, Sul79, Sul84, Nic89, BJ97, McM99, Pei10]. One of the main result of these papers is the relation between δ and the hausdorff dimension of the limit set. The limit set of a group $\Gamma \curvearrowright X$ is the set Λ defined by $\Lambda := \overline{\Gamma \cdot o} \setminus \Gamma \cdot o \subset \partial X$. The boundary of \mathbb{H}^n is the sphere $\partial\mathbb{H}^n = \mathbb{S}^{n-1}$, the limit set does not depend of the base point $o \in \mathbb{H}^n$. For example the limit set of a cocompact group is the whole boundary. We will not give a precise definition of the Hausdorff dimension. Let us still give a very rough (and not exact) definition : let $N(r)$ be the number of balls of radius r that are necessary to cover a subset A of a metric space, then when r is very small $N(r)$ behaves more or less as $1/r^\alpha$ where α is the Hausdorff dimension of A . The most evolved result in this direction [BJ97] says that for discrete non-elementary isometry group Γ of \mathbb{H}^n (ie. $\text{Card } \Lambda_\Gamma = +\infty$), the critical exponent $\delta(\Gamma \curvearrowright \mathbb{H}^3)$ is equal to the Hausdorff dimension of the conical limit set. Without being precise the conical limit set is a subset of Λ , whose points are well approached by elements of the group.

Another classical question is what can we say of the critical exponent of a subgroup of Γ . M. Peigné [Pei10] showed that if $H < \Gamma$ is such that $\Lambda_H \neq \Lambda_\Gamma$ and if the Poincaré series associated to Γ diverges at $\delta(\Gamma)$, then $\delta(H) < \delta(\Gamma)$. However, if $H \triangleleft \Gamma$ is a normal subgroup of Γ , $\Lambda_H = \Lambda_\Gamma$ and we cannot say much. A result of Falk and Stratmann [FS04] says that $\delta(H) \geq \frac{\delta(\Gamma)}{2}$, and if the Poincaré series associated to Γ diverges at $\delta(\Gamma)$ then the inequality is strict $\delta(H) > \frac{\delta(\Gamma)}{2}$ [Rob05].

The fact that the divergence of the Poincaré series appears as a hypothesis is due to the use of Patterson-Sullivan theory of conformal density on the boundary, which works in the divergence case. A precise statement of this remark can be found in the book of T. Roblin [Rob03].

Let us say a few words on an example at the edge of the two way to generalize the study of critical exponent. Let Γ be a surface group, that is the fundamental group of a compact surface of genus at least two. By the uniformization Theorem, there are discrete and faithful representations of Γ into $\text{Isom}^+(\mathbb{H}^2)$. These representations are called Teichmüller representations. The set of Teichmüller representations up to conjugacy is called the Teichmüller space of S and denoted by $\text{Teich}(S)$. As we remarked, in that case $\delta(\Gamma \curvearrowright \mathbb{H}^2) = 1$. However, \mathbb{H}^2 embedded as a totally geodesic subspace of \mathbb{H}^3 , and the isometry group of \mathbb{H}^2 is a subgroup of $\text{Isom}(\mathbb{H}^3)$. A Teichmüller representation $\rho_0 : \Gamma \rightarrow \text{Isom}(\mathbb{H}^2)$ can then be seen as a representation in $\text{Isom}(\mathbb{H}^3)$, $\rho : \Gamma \rightarrow \text{Isom}(\mathbb{H}^2) \rightarrow \text{Isom}(\mathbb{H}^3)$. In this case, it is called Fuchsian. Its critical exponent $\delta(\Gamma \curvearrowright \mathbb{H}^3)$ is still equal to 1, since \mathbb{H}^2 is totally geodesic in \mathbb{H}^3 . But in the group $\text{Isom}(\mathbb{H}^3)$ we can modify the representation such that it does not factor anymore by $\text{Isom}(\mathbb{H}^2)$. These representations are called quasi-Fuchsian representations and we have the following rigidity result due to R. Bowen

Theorem 2.3. [Bow79] $\delta(\Gamma \curvearrowright \mathbb{H}^3) \geq 1$ with equality if and only if the representation is Fuchsian.

Critical exponent of quasi-Fuchsian representations has also been intensively studied. L. Bers exhibited a parametrization of the set of quasi-Fuchsian representations by the product of two Teichmüller spaces[Ber60]. Many authors have been interested by the relation between the critical exponent of the quasi-Fuchsian representation and the two Teichmüller representations coming from the Bers parametrization. For example, J. Brock [Bro03] looked at large scale behaviour. He proved that the critical exponent is closed to 2 if and only if the two underlying hyperbolic surfaces coming from the two Teichmüller representations are far away (for the Weil-petterson distance).

C. McMullen [McM99] studied the behaviour of critical exponent in various examples when one of the two surfaces goes to infinity in $\text{Teich}(S)$.

A. Sanders [San14], studied the behaviour of critical exponent near the Fuchsian locus, and showed an isolation theorem for the critical exponent.

Finally, let us give a famous result where the group is fixed and we play with the space. Let M be a compact hyperbolic manifold of dimension $n \geq 3$. Let Γ be its fundamental group. Then G. Besson, G. Courtois and S. Gallot [BCG95] showed that for any other Riemannian simply connected manifold X such that $\Gamma \curvearrowright X$ cocompactly, then $\text{Vol}(X/\Gamma)\delta(\Gamma \curvearrowright X)^n \geq \text{Vol}(M)\delta(\Gamma \curvearrowright \mathbb{H}^n)^n$, with equality if and only if X is homothetic to \mathbb{H}^3 .

2.3. Diagonal action on $\mathbb{H}^2 \times \mathbb{H}^2$. This paper intends to study the critical exponent for the diagonal action of two Teichmüller representations.

Let ρ_1, ρ_2 be two Teichmüller representations, they acts on $\mathbb{H}^2 \times \mathbb{H}^2$ by the diagonal action : $\forall \gamma \in \Gamma, \forall x = (x_1, x_2) \in \mathbb{H}^2 \times \mathbb{H}^2$:

$$\gamma \cdot x = (\rho_1(\gamma)x_1, \rho_2(\gamma)x_2).$$

Since $\mathbb{H}^2 \times \mathbb{H}^2/\Gamma$ is topologically a \mathbb{H}^2 bundle over S , this action is not cocompact.

In order to define the critical exponent of $\Gamma \curvearrowright \mathbb{H}^2 \times \mathbb{H}^2$, we need to choose a metric on $\mathbb{H}^2 \times \mathbb{H}^2$. We will use the Manhattan metric, which is defined by $d((x_1, x_2), (y_1, y_2)) = d_{\mathbb{H}}(x_1, y_1) + d_{\mathbb{H}}(x_2, y_2)$. Since the critical exponent does not depend on the base point, it neither depends on the conjugacy class of the representation, hence it defines a function on the product of the Teichmüller spaces. The choice of the Manhattan metric is motivated for two reasons.

The first one is the simplicity of Busemann compactification for this metric, compare to the Riemannian metric, which is useful for Patterson-Sullivan theory, as in [Bur93].

The second motivation for the choice of Manhattan metric is its link with Anti-de Sitter geometry. Indeed, even though Anti-de Sitter is not a metric space, we can define a distance between some pairs of points. For *globally hyperbolic* Anti-de Sitter manifold, the natural critical exponent that arises with this definition, coincides (up to a factor 2) with the critical exponent on $\mathbb{H}^2 \times \mathbb{H}^2$ endowed with the Manhattan metric. I will give more details on this link in section 5.

From now on $\delta(\cdot, \cdot)$ will stand for the function defined on $\mathbb{H}^2 \times \mathbb{H}^2$ by $\delta((\rho_1, \rho_2) \curvearrowright \mathbb{H}^2 \times \mathbb{H}^2)$, that is the critical exponent for

$$P(S_1, S_2; s) = \sum_{\gamma \in \Gamma} e^{-s(d(\rho_1(\gamma) \cdot x, x) + d(\rho_2(\gamma) \cdot x, x))},$$

where $S_i := \mathbb{H}^2/\rho_i(\Gamma)$, $i \in \{1, 2\}$.

There was a series of papers during the 90's which studied this critical exponent or some related quantities, by C. Bishop, T. Steeger, M. Burger, R. Schwartz and R. Sharp [BS91, Bur93, SS93, Sha98]. More generally critical exponent and Patterson-Sullivan theory for product of Hadamard spaces has been studied latter by J-F. Quint, G. Link, F. Dal'bo and I. Kim [Qui02, Lin04, DK06, DK08].

The first cited series of papers obtained similar result as the quasi-Fuchsian setting. For example, it is proven in [BS91] that $\delta(S, S') \geq 1/2$ with equality if and only if $S = S'$ in the Teichmüller space of S . In [Bur93] there is a result which related the critical exponent to the Hausdorff dimension of a subset of the limit set. Finally in [SS93, Sha98] critical exponent is related to the repartition of the number of closed geodesics.

I followed this path and looked at the behaviour of $\delta(\cdot, \cdot)$ when the parameters range over the Teichmüller space. I obtained the counter part of the examples of C. McMullen, and of the isolation result of A. Sanders. The next two sections are devoted to explain these results.

Before starting to exhibit examples, let us explain how critical exponent can be interpreted geometrically. As we said, the critical exponent δ does not depend on the conjugacy classes of ρ_1 and ρ_2 . Let $S_i := \mathbb{H}^2/\Gamma_i$, $i \in \{1, 2\}$ be the underlying marked hyperbolic surfaces homomorphic to S . Let \mathcal{C} be the set of free homotopy classes of closed curves on S . Since S_i are hyperbolic surfaces, for every $c \in \mathcal{C}$, there is a unique geodesic in its homotopy class. The length of this geodesic will be denoted by $\ell_i(c)$. Let us insist on the fact that S_i are *marked* surfaces, meaning they come with a homomorphism $f_i : S \rightarrow S_i$, or equivalently we keep the knowledge of the image of the generators of Γ by the representations. This allow to know which curve is send on which one.

Let us explain the difference in the genus 1 case. The fundamental group of a torus is isomorphic to $\Gamma = \mathbb{Z}^2$ generated by a meridian a and a longitude b . Now the two representations

$$\rho_1 \begin{cases} \Gamma & \longrightarrow \text{Isom}(\mathbb{R}^2) \\ (a, b) & \longmapsto (t_{(1,0)}, t_{(0,1)}) \end{cases}$$

and

$$\rho_2 \begin{cases} \Gamma & \longrightarrow \text{Isom}(\mathbb{R}^2) \\ (a, b) & \longmapsto (t_{(1,0)}, t_{(1,1)}) \end{cases} ,$$

where $\tau_{(a,b)}$ is the translation by the vector (a, b) . Those two representations give two isometric torus, but the curve b has length 1 in $T_1 := \mathbb{R}^2/\rho_1(\Gamma)$, and length $\sqrt{2}$ in $T_2 := \mathbb{R}^2/\rho_2(\Gamma)$.

In my Ph.D thesis I showed using similar arguments as in [Kni83] that the critical exponent $\delta(S_1, S_2)$ is equal to the exponential growth of the number of closed geodesics in the pair " $(S_1 + S_2)$ ". By this we mean :

$$\delta(S_1, S_2) = \limsup_{R \rightarrow \infty} \frac{1}{R} \log \text{Card} \{c \in \mathcal{C} \mid \ell_1(c) + \ell_2(c) \leq R\} .$$

Therefore we can see δ has a measure of how the geodesics on S_1 and S_2 are different from each other. This will be our point of view from now on.

3. EXAMPLES OF BEHAVIOUR

This section is devoted to the asymptotic behaviour of $\delta(\cdot, \cdot)$ on the product of Teichmüller spaces on some particular cases.

3.1. Iteration of Mapping class group. The mapping class group MCG of S is the space of diffeomorphisms of S modulo the following relation : $f \sim g$ if and only if $f \circ g^{-1}$ is isotopic to the identity. It acts on *marked* hyperbolic surfaces, by changing the marking. Elements of MCG are classified in three types : finite order, reducible, or pseudo-Anosov. Finite order ones are not interesting for us, since their orbits are bounded.

3.1.1. Dehn twist. Dehn twists are diffeomorphism which makes a full turn along a simple closed curve. They can be generalized to countinuous time parameter and are then called Fenchel-Nielsen twists (they are not elements of the mapping class group). We will study the behaviour of δ along Fenchel-Nielsen twists in section 3.4. These Fenchel-Nielsen twists can also be generalized by twisting along measured geodesic laminations instead of simple closed curves. These transformations are named after Thurston terminology *earthquakes*. A geodesic lamination is a closed subset of S foliated by geodesics. A measured geodesic lamination is a geodesic lamination endowed with a transverse measure, invariant by the holonomy consisting of moving along the leaves. A simple closed curve can be seen as a measured geodesic lamination by considering the Dirac measure on it.

Finally we will also use the notion of *geodesic currents* [Bon86, Bon88]. Roughly, the set of geodesic currents is a completion for the intersection form of the set of all closed geodesics. We

will denote it by $\text{Curr}(S)$. With this terminology, the set of measured geodesic laminations is identified with the set of geodesic currents of 0 self intersection.

Proposition 3.1. *Let α be a simple closed curve on S and τ be the Dehn twist along α . Then*

$$n \rightarrow \delta(\tau^{2n}S, S),$$

is decreasing in n . In particular $\lim_{n \rightarrow \infty} \delta(\tau^{2n}S, S)$ exists and is strictly less than $1/2$.

The idea for this simple case is to "symmetrize" the function using precomposition by τ^{-n} :

$$\delta(\tau^{2n}S, S) = \delta(\tau^n S, \tau^{-n}S).$$

This relation uses intrinsically that τ is an element of the mapping class group, it comes from a change of variable in the Poincaré series, that can only be done since the mapping class group acts as change of markings.

Then we can show using the convexity of length along Fenchel-Nielsen twist (or more generally earthquakes [Ker83]), that

$$n \rightarrow \ell(\tau^n c) + \ell(\tau^{-n} c),$$

is increasing. Which in turns implies that the Poincaré series $P_n(s) = \sum_{c \in \mathcal{C}} e^{-s(\ell(\tau^n c) + \ell(\tau^{-n} c))}$ is a decreasing sequence of functions, which by definition says that $\delta(\tau^n S, \tau^{-n} S)$ is decreasing.

3.1.2. Pseudo Anosov. A very useful notion to estimate the length of a curve is the intersection. The geometric intersection between two curves, α, β is the number of intersections of their geodesic representatives. We denote it by $i(\alpha, \beta)$. We say that a closed curve or a union of closed curves α *fills* the surface if for every geodesics c we have $i(\alpha, c) > 0$. As we said, this notion generalizes to the set of geodesic currents, see [Bon88] for more details. One of the nice property of a curve α (or current) that fills S is that the projective set of curves that intersects α a bounded number of time is compact [Bon88]. From this we can deduce

Lemma 3.2. *Let α be a union of closed curves (or a geodesic current) that fills S . Then there is $K > 0$ such that for all closed curves c (for all geodesic currents)*

$$i(\alpha, c) \geq K \ell(c).$$

An example of geodesics which fills S is the union of two transverse decomposition of S in pair of pants. A non-example is a geodesic lamination since they have 0 self intersection.

We recall the definition of pseudo-Anosov diffeomorphism.

Definition 3.3. [CT07, Section 13] *A diffeomorphism $A : S \rightarrow S$ is said pseudo-Anosov if there exists two measured geodesic laminations (\mathcal{L}_+, μ_+) , (\mathcal{L}_-, μ_-) , one transverses to the others, and a constant $k > 1$ such that :*

- (1) $\mathcal{L}_+ \cup \mathcal{L}_-$ fills S .
- (2) $A(\mathcal{L}_\pm) = \mathcal{L}_\pm$.
- (3) $A^* \mu_+ = k \mu_+$. (inverse image of μ_+ by A)
- (4) $A^* \mu_- = \frac{1}{k} \mu_-$. (inverse image of μ_- by A)

Definition 3.4. *Let R be a subsurface of S , with geodesic boundary. We say that a diffeomorphism A is R -pseudo-Anosov, if A is the identity on $S \setminus R$ and pseudo-Anosov on R .*

Remark that a S -pseudo-Anosov is a pseudo-Anosov diffeomorphism.

Proposition 3.5. *Let R be a subsurface of S with geodesic boundary (possibly $R = S$). Let A be a R -pseudo-Anosov diffeomorphism. Then*

$$\lim_{n \rightarrow \infty} \delta(A^n S, S) = \delta(S \setminus R, S \setminus R).$$

We showed this proposition for $R = S$ in [Glo15]. The proof follows the same lines in this case. Since we iterate an element of the mapping class group, we can symmetrize $\delta(A^{2n}S, S) = \delta(A^nS, A^{-n}S)$ as we did for Dehn twists to obtain global behaviour on every curves. Then from the definition of R -pseudo-Anosov diffeomorphism we can estimate the lengths of geodesics for the pair $(A^nS, A^{-n}S)$:

$$\ell(A^n c) + \ell(A^{-n} c) \geq Ci(\mu_+ + \mu_-, c)\lambda^n.$$

Since (μ_+, μ_-) are the laminations of a R -pseudo-Anosov : $i(\mu_+ + \mu_-, c) \geq C\ell_R(c)$. Hence if $\ell_R(c) \neq 0$, $\ell(A^n c) + \ell(A^{-n} c)$ tends to infinity uniformly. If $\ell_R(c) = 0$ then the R -pseudo-Anosov does not act on c . We divide the Poincaré series :

$$\begin{aligned} \sum_{c \in \mathcal{C}} e^{-s(\ell(A^n c) + \ell(A^{-n} c))} &= \sum_{c, i(c, R) \neq 0} e^{-s(\ell(A^n c) + \ell(A^{-n} c))} + \sum_{c, i(c, R) = 0} e^{-s(\ell(A^n c) + \ell(A^{-n} c))} \\ &\leq \sum_{c, i(c, R) \neq 0} e^{-sK\lambda^n \ell_R(c)} + \sum_{c, i(c, R) = 0} e^{-s(\ell(c) + \ell(c))} \end{aligned}$$

The critical exponent for the first sum of the right hand side tends to 0. The critical exponent for the second sum of the right hand side is equal to $\delta(S \setminus R, S \setminus R)$. The result follows.

Corollary 3.6. *Let $X : \mathbb{R}^+ \rightarrow \text{Teich}(S)$ be a Teichmüller geodesic ray whose endpoint is a fixed point of a pseudo-Anosov diffeomorphism. Then*

$$\lim \delta(X(t), S) = 0.$$

Proof. Let A be the pseudo-Anosov whose attractive fixed point is the endpoint of X . From the previous result $\delta(A^n S, S) \rightarrow 0$. Let S_0 be a point in $\text{Teich}(S)$ on the geodesic axis of A , and let us call $A : \mathbb{R} \rightarrow \text{Teich}(S)$ this geodesic axis parametrized in such a way that $A^n S_0 = A(n)$. Remark that by a simple calculation we have

$$(2) \quad \delta(X, Y) \leq \exp(\max(d_L(X, X'), d_L(Y, Y')))\delta(X', Y'),$$

where d_L is the Lipschitz asymmetric distance, introduced by W. Thurston [Thu98] and defined by $d_L(X, X') := \log \sup_{c \in \mathcal{C}} \frac{\ell_{X'}(c)}{\ell_X(c)}$. It follows that

$$\delta(A(t), S_0) \leq \exp(d_L(A[t], A(t)))\delta(A[t], S_0),$$

where $[t]$ is the floor of t .

A classical result following from the work of S. Kerchoff and S. Wolpert [Ker80, Wol79] says that the lipschitz distance is smaller than the Teichmüller distance, therefore

$$\delta(A(t), S_0) \leq \exp(\lambda|[t] - t|)\delta(A^{[t]}S_0, S_0).$$

Here λ is the translation distance of the pseudo-Anosov diffeomorphism, it appears because of our parametrization : $d_T(A(0), A(1)) = \lambda$.

Since $\delta(A^{[t]}S_0, S_0) \rightarrow 0$ it follows that $\delta(A(t), S_0) \rightarrow 0$.

Applying again Equation (2) we have that for all S

$$\delta(A(t), S) \leq \exp(d_L(S, S_0))\delta(A[t], S_0).$$

Once more, it implies that $\delta(A(t), S) \rightarrow 0$.

Finally, we used a deep result of H. Masur, [Mas80] classifying Teichmüller rays staying at bounded distances. Since the stable foliation of a pseudo-Anosov is uniquely ergodic, it follows that any geodesic ray $X : \mathbb{R}^+ \rightarrow \text{Teich}(S)$ whose endpoint is equal to $A(+\infty)$, stay at bounded Teichmüller distance to A . Using one more time the estimates of Kerchoff and Wolpert, and the Equation (2), we conclude there exists $K > 0$ such that

$$\delta(X(t), S) \leq K\delta(A(t), S).$$

□

3.2. Random walk on mapping class group. We present in this section some results on the behaviour of critical exponent along random walks paths. Let ν be a probability measure on $MCG(S)$ of finite first moment (with respect to Teichmüller distance) and whose support generates a non-elementary subgroup. We consider the random element $f_n = g_n g_{n-1} \dots g_1$ where g_i are chosen independently and distributed with ν . Then we can look at the random path $f_n S \in \text{Teich}(S)$. From the work of V. Kaimanovich and H. Masur, this path a.s. converges to a point of uniquely ergodic measure foliation on the Thurston boundary of Teichmüller space. We want to study the behaviour of $\delta(f_n S, S)$ and of $\delta(f_n S, f'_n S)$ where f'_n is another random path independent of f_n .

Using the Busemann compactification of C. Walsh [Wal10], A. Karlsson proved the following

Theorem 3.7. [Kar14] *There exists a (random) measured foliation μ and a constant C depending on S and μ such that for any $\epsilon > 0$ there is a number N for which*

$$Ci(\mu, c)(\lambda - \epsilon)^n \leq \ell_S(f_n c) \leq \ell(\lambda + \epsilon)^n,$$

holds for all simple closed curves c and any $n > N$.

In fact, it is possible using his proof to show that it holds for any closed curve (any geodesic current) c , not necessarily simple. Let us explain why. As we just said, the proof of A. Karlsson relies on the fact that the random walk converges in the Busemann compactification defines by C. Walsh. Walsh defined a horofunction of the (projective) class μ by

$$\Psi_\mu(x) = \log \left(\sup_{\eta \in ML} \frac{i(\mu, \eta)}{\ell_x(\eta)} / \sup_{\eta \in ML} \frac{i(\mu, \eta)}{\ell_b(\eta)} \right).$$

In this formula, ML stands for the set of measured geodesic laminations. $\ell_*(\eta)$ is the length of the measured lamination in the hyperbolic surface. For our purpose there is no need to know the exact definition, let us say that for a measured geodesic lamination which is a simple closed curve, this is equal to its geodesic length and it is continuous on $\text{Teich}(S) \times ML$. Finally $b \in \text{Teich}(S)$ is a base point.

Let d_L be the Lipschitz (or Thurston) asymmetric distance. This is often defined as we previously did by the following formula :

$$d_L(x, y) := \log \sup_{\eta \in ML} \frac{\ell_y(\eta)}{\ell_x(\eta)}.$$

However this is *not* the original definition in [Thu98]. Indeed, in this article W. Thurston defined d_L by

$$d_L(x, y) = \inf_{\phi \simeq Id} (\log(L(\phi))),$$

where the supremum is taken over all Lipschitz map ϕ between x and y isotopic to the identity and $L(\phi)$ is the Lipschitz constant defined by $L(\phi) = \sup_{u \neq v} \left(\frac{d_y(\phi(u), \phi(v))}{d_x(u, v)} \right)$. He then showed that

$$d_L(x, y) := \log \sup_{c \in \mathcal{C}} \frac{\ell_y(\eta)}{\ell_x(\eta)},$$

where the supremum is taken over \mathcal{C} the set of *all* closed curves. He finally proved that the supremum is attained along geodesic laminations. That is

$$\sup_{c \in \mathcal{C}} \frac{\ell_y(\eta)}{\ell_x(\eta)} = \sup_{\eta \in ML} \frac{\ell_y(\eta)}{\ell_x(\eta)}.$$

Now we define

$$\widetilde{\Psi}_\mu(x) := \log \left(\sup_{\eta \in \mathcal{C}} \frac{i(\mu, \eta)}{\ell_x(\eta)} / \sup_{\eta \in \mathcal{C}} \frac{i(\mu, \eta)}{\ell_b(\eta)} \right),$$

where this time the supremum is taken over all closed curves. By the work of F. Bonahon, the set of closed curves is dense in $Curr(S)$ the set of geodesic currents on S , we hence have also

$$\widetilde{\Psi}_\mu(x) = \log \left(\sup_{\eta \in Curr(S)} \frac{i(\mu, \eta)}{\ell_x(\eta)} / \sup_{\eta \in Curr(S)} \frac{i(\mu, \eta)}{\ell_b(\eta)} \right).$$

From the previous remarks, when $\mu = z$ correspond to the Liouville current associated to a point of the Teichmüller space, the two functions coincides :

$$\widetilde{\Psi}_\mu(x) = \Psi_\mu(x) = d_L(x, z) - d_L(b, z).$$

C. Walsh proved that the function $z \rightarrow \Psi_z$ is continuous in the Thurston compactification of the Teichmüller space. In his proof, he used the compactness of $\{\eta \in ML \mid \ell_b(\eta) \leq 1\}$ which can be extended to geodesic current by the work of F. Bonahon : $\{c \in Curr(S) \mid \ell_b(c) \leq 1\}$ is also compact in $Curr(S)$. Therefore his proof can be followed step-by-step to show that the map $\widetilde{\Psi} : \overline{Teich(S)} \rightarrow C(Teich(S)) : z \mapsto \widetilde{\Psi}_z$ is continuous.

Hence $\widetilde{\Psi}_\mu$ also coincides with Ψ_μ on the boundary of the Teichmüller space. In other words :

$$\Psi_\mu(x) = \log \left(\sup_{\eta \in \mathcal{C}} \frac{i(\mu, \eta)}{\ell_x(\eta)} / \sup_{\eta \in \mathcal{C}} \frac{i(\mu, \eta)}{\ell_b(\eta)} \right).$$

Then we follow the proof of A. Karlsson and find the more general statement :

Theorem 3.8. *There exists a (random) measured foliation μ and a constant C depending on S and μ such that for any $\epsilon > 0$ there is a number N for which*

$$Ci(\mu, c)(\lambda - \epsilon)^n \leq \ell_S(f_n c) \leq \ell(\lambda + \epsilon)^n,$$

holds for all closed curves (all geodesic currents) c and any $n > N$.

Proposition 3.9. *For a.e. independent random paths $f_n = g_n g_{n-1} \dots g_1$ and $f'_n = g'_n g'_{n-1} \dots g'_1$,*

$$\delta(f_n S, f'_n S) \rightarrow 0.$$

Proof. Using 3.8, we have that for all curves and n big enough that

$$\ell(f_n c) \geq Ci(\mu, c)(\lambda - \epsilon)^n.$$

$$\ell(f'_n c) \geq C'i(\mu', c)(\lambda' - \epsilon)^n.$$

Let $C'' = \min(C, C')$ and $\lambda'' = \min(\lambda, \lambda')$. We then have

$$\ell(f_n c) + \ell(f'_n c) \geq C''i(\mu + \mu', c)(\lambda'' - \epsilon)^n.$$

From the work of Kaimanovich Masur μ and μ' are uniquely ergodic, in particular it implies that if a geodesic α , satisfies $i(\mu, \alpha) = 0$ then α is in the support of μ . Moreover a.s. $\mu \neq \mu'$ since the Poisson boundary is non atomic. Hence if a geodesic α satisfies $i(\mu, \alpha) = 0$ then $i(\mu', \alpha) > 0$. In particular, $\mu + \mu'$ satisfies the condition of lemma 3.2. There exists $K > 0$ such that

$$\ell(f_n c) + \ell(f'_n c) \geq C''K\ell(c)(\lambda'' - \epsilon)^n.$$

□

3.3. Pinching curves.

Proposition 3.10. *Let P, P' be two transverse pair of pants decompositions of S . Let S_n, S'_n be the sequences of surfaces obtain by pinching P and P' by a factor $\epsilon_n \rightarrow 0$. Then*

$$\lim_{n \rightarrow \infty} \delta(S_n, S'_n) = 0$$

The so-called "collar lemma" said that on a hyperbolic surface when a curve c intersects a very small curve then it has to be very long [Hal81]. So we can show that there exists $C > 0$ such that for all closed curves c

$$\ell_{S_n}(c) + \ell_{S'_n}(c) \geq C |\log(\epsilon_n)| i(c, P \cup P')$$

Then since $P \cup P'$ fills the surface we can conclude by lemma 3.2.

This previous example gives an obstruction to produce a distance directly from δ with nice properties.

Proposition 3.11. *There is no function $f :]0, 1/2] \rightarrow \mathbb{R}^+$ such that*

- *f is continuous.*
- *$\lim_{x \rightarrow 0} f(x) = +\infty$*
- *The function d defined on $\text{Teich}(S) \times \text{Teich}(S)$ by $d(S_0, S_1) = f(\delta(S_0, S_1))$ satisfies the triangular inequality.*

Indeed suppose by contradiction there is such a distance. We can show, since there exists some curves whose length does not go to infinity that $\delta(S, S_n)$ does not go to 0 (take any pair of curves on S which generate a free group in the fundamental group and whose length is bounded in S_n , then the critical exponent associated to these free groups is strictly bigger than $\epsilon > 0$). The same applies to $\delta(S, S'_n)$. Since f is continuous, we have that there exists $K > 0$ such that $d(S, S_n) < K$ and $d(S, S'_n) < K$. But $\delta(S_n, S'_n) \rightarrow 0$ hence by the second hypothesis on f we have $d(S_n, S'_n) \rightarrow \infty$. Hence it cannot satisfy the triangular inequality.

If we pinch the same set of geodesics then the previous behaviour does not happen. In fact, this is the only example, we know where the critical exponent goes to 1/2 although the surfaces goes to infinity in $\text{Teich}(S)$.

Proposition 3.12. *Let ϵ_n and ϵ'_n be two sequences going to 0. Let S_n and S'_n be the surfaces obtained by pinching P of a factor ϵ_n and ϵ'_n respectively. Then*

$$\lim_{n \rightarrow \infty} \delta(S_n, S'_n) = 1/2.$$

This follows from the fact that Weil-Petersson distance between S_n and S'_n goes to 0 and the relation between Weil-Petersson distance and the intersection of two Liouville currents.

3.4. Fenchel Nielsen twists. This example is the basic idea for the isolation Theorem that we will explain in section 4. Let α be a simple closed curve and $\mathcal{E}_\alpha^t()$ the Fenchel-Nielsen twist along α . This is a time continuous version of the Dehn twist around α , ie $\tau := \mathcal{E}_\alpha^{\ell_0(\alpha)}()$ is the Dehn twist around α . Let S_0 be a hyperbolic surface and defined $S_t := \mathcal{E}_\alpha^t(S_0)$. Fix $t \in [0, 2\ell_0(\alpha))$, then as the first example we can show that $\delta(S_0, S_{2n+t}) = \delta(\tau^{-n}S_0, \tau^n S_t)$ is decreasing, using convexity of length function. Hence the following limit exists :

$$\delta(t) := \lim_{n \rightarrow \infty} \delta(S_0, S_{2n+t}).$$

It is clear that $t \rightarrow \delta(t)$ is $2\ell_0(\alpha)$ -periodic. Moreover we can show, using one more time convexity of length function along Fenchel-Nielsen twists and a compactness argument of "bounded length" geodesic currents, that

Proposition 3.13. *The function $t \mapsto \delta(S_0, S_t)$ is uniformly continuous.*

This shows in one hand that $t \mapsto \delta(t)$ is continuous and in other hand that $\lim_{t \rightarrow \infty} |\delta(S_0, S_t) - \delta(t)| = 0$.

It is clear from the first example on Dehn twists that $\delta(t) < 1/2$. Then we have

Corollary 3.14. *For any Fenchel-Nielsen twist along a simple closed curve :*

$$\limsup \delta(S_0, S_t) < 1/2.$$

4. ISOLATION THEOREM

We exhibited various examples where we can estimate the limit of $\delta(S_n, S)$ as $S_n \rightarrow \partial \text{Teich}(S)$. Noticing that the limit is never equal to $1/2$, it is natural to ask if it is always the case. We answered by the affirmative at this question.

Theorem 4.1. *Let S_n be a sequence in $\text{Teich}(S)$ going to infinity. Then*

$$\limsup_{n \rightarrow \infty} \delta(S_n, S) < 1/2.$$

or equivalently

Theorem (reformulation of 3.1). *Let S_n be a sequence in $\text{Teich}(S)$. Then*

$$\lim \delta(S_n, S) = 1/2 \quad \text{iff} \quad \lim S_n = S \text{ in } \text{Teich}(S)$$

Even if the result seems to be very natural, the problem is there is no easy way to reach the boundary knowing the length of all closed curves. From the previous examples, we see there are (at least) two natural ways to go to infinity on $\text{Teich}(S)$: following a Teichmüller geodesic ray or following an earthquake. In our proof we chose the second one, since earthquakes are directly connected to the hyperbolic geometry.

Since every points of the Teichmüller space is attained by an earthquake, theorem 4.1 is a consequence of

Theorem 4.2. *Let S_t be an earthquake path in $\text{Teich}(S)$ then*

$$\limsup_{t \rightarrow \infty} \delta(S_t, S) < 1/2.$$

Here again, for our purpose we don't need to know the formal definition of an earthquake. As we said, earthquakes are generalisation to any measured laminations of Fenchel-Nielsen twist along simple closed curves. We need to prove the same result as corollary 3.14 *without* the possibility of using the "symmetrization" trick.

For this we have shown in [Glo15] that along any laminations \mathcal{L} , the length of "most" of the curves are increasing along the earthquake directed by \mathcal{L} : just think of Fenchel Nielsen twist along a very long curve. All the subtlety lies in what we mean by "most".

Definition 4.3. *Let S be a hyperbolic surface. We say that a property P is satisfied by most curves on S if there is $\eta > 0$ such that*

$$\frac{\text{Card}\{c \in \mathcal{C} \mid \ell_S(c) \leq R\} \cap P}{\text{Card}\{c \in \mathcal{C} \mid \ell_S(c) \leq R\}} = o(e^{-\eta R}).$$

Let S, S' be two hyperbolic surfaces. We say that a property P is satisfied by most curves on $S + S'$ if there is $\eta > 0$ such that

$$\frac{\text{Card}\{c \in \mathcal{C} \mid \ell_S(c) + \ell_{S'}(c) \leq R\} \cap P}{\text{Card}\{c \in \mathcal{C} \mid \ell_S(c) + \ell_{S'}(c) \leq R\}} = o(e^{-\eta R}).$$

A theorem of Y. Kifer [Kif94] on large deviation of geodesic flow allows us to show that a property is satisfied by most of the curve on *one* surface. His theorem does probably not generalise to the "sum" of two surfaces, since $\mathbb{H}^2 \times \mathbb{H}^2$ is not hyperbolic.

However, in this terminology, we showed that most of the curves on S are increasing along any earthquakes. This part of the proof uses many different arguments : compactness of geodesic currents of bounded length, compactness of projective measure geodesic laminations, convexity of length along earthquakes, and a rigidity result on intersection between two Liouville currents.

The next problem to deal with is that δ is defined through the use of the Manhattan metric, hence we need a result for most of the curves on $S + S_n$. A result of G. Link [Lin04, Theorem 3.12, Theorem 5.1] or J.-F. Quint [Qui02] (more precisely we proved its counterpart for the Manhattan metric) showed there is a λ_n for which most of the curves on $S + S_n$, satisfy $\ell_S(c)/\ell_{S_n}(c)$ almost equal to λ_n . Moreover I showed using the Manhattan curve defined in [Bur93], that if $\delta(S, S_n) \rightarrow 1/2$ then $\lambda_n \rightarrow 1$.

Putting everything together, it can then be shown that there is $\eta > 0$ such that

$$(3) \quad \delta(S, S_n) \leq \frac{1 - \eta}{1 + \lambda_n}.$$

Indeed, in the Poincaré séries associated to $(S + S_n)$ we can take out the set of curves whose length increases uniformly, since their associated critical exponent tends to 0. Then, there are only "few" curves on S in the complementary set of these curves : that is where the $1 - \eta$ comes from. Moreover, in these complementary set, most of the curve for $S + S_n$ satisfies $\ell_S(c) + \ell_{S_n}(c) \simeq (1 + \lambda_n)\ell_S(c)$ and that is where $\frac{1}{1 + \lambda_n}$ comes from.

We conclude by contradiction. Supposing that $\delta \rightarrow 1/2$, Hence passing to the limit in equation 3 gives $\frac{1}{2} \leq \frac{1 - \eta}{2}$ since $\lambda_n \rightarrow 1$.

5. ANTI-DE SITTER INTERPRETATION

It appears that this critical exponent can be related to Anti-de Sitter geometry. The Anti-de Sitter space AdS is a Lorentzian manifold of constant negative curvature, so it is the Lorentzian counterpart of the hyperbolic space. A nice model is to consider $\mathrm{PSL}_2(\mathbb{R})$ endowed with its Killing form. In this model the identity component of the isometry group can be identified to $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ acting by left and right multiplication.

A class of manifolds modelled on AdS are the so-called *globally hyperbolic*. These manifolds are topologically the product of $S \times (0, 1)$ and are the counterpart of Quasi-Fuchsian manifold in the hyperbolic setting. For our purpose, the main result to know is a result of G. Mess [Mes07] : in dimension 3, globally hyperbolic manifolds with base surface S are parametrized by the product of two Teichmüller spaces. Given two Teichmüller representations, then there exists a (maximal) convex subset of AdS^3 , called "black domain", where the action is properly discontinuous, the quotient is a globally hyperbolic manifold. We called Fuchsian the globally hyperbolic manifolds which are parametrized by the two same points of $\mathrm{Teich}(S)$, in that case the black domain contains a totally geodesic copy of the hyperbolic plane.

I proved that on a globally hyperbolic AdS^3 manifold, in every isotopy class of closed curve there is a unique geodesic. This geodesic is space-like, meaning that the quadratic form on its tangent vector is positive. We define the length of this geodesic by the Riemannian length for the induce metric. Remark that since the manifold is not Riemannian, this length does neither minimize nor maximize the integral $\int g(\dot{c}(t), \dot{c}(t))dt$ on the isotopy class of the curve. However a simple calculation shows that for a closed curve c in a globally hyperbolic manifold parametrized by (S_1, S_2) , its length in the globally hyperbolic manifold $\ell_{\mathrm{AdS}}(c)$, is given by

$$\ell_{\mathrm{AdS}}(c) = \frac{1}{2}(\ell_{S_1}(c) + \ell_{S_2}(c)).$$

We then define the critical exponent of a globally hyperbolic AdS^3 manifold, M as

$$\delta(M) := \limsup_{R \rightarrow \infty} \frac{1}{R} \log \text{Card} \{c \in \mathcal{C} \mid \ell_{\text{AdS}}(c) \leq R\}.$$

The previous remark says that if M is parametrized by (S_1, S_2) then $\delta(M) = 2\delta(S_1, S_2)$.

Hence the previous results on Teichmüller representations acting on $\mathbb{H}^2 \times \mathbb{H}^2$ give informations on the dynamics of globally hyperbolic manifolds. We will not translate all previous statements in this vocabulary, but we just want to mention that the Bishop Steger theorem [BS91] in this setting gives the counterpart of Bowen's theorem for quasi-Fuchsian manifolds :

Theorem 5.1. *Let M be a globally hyperbolic AdS^3 manifold. Then*

$$\delta(M) \geq 1,$$

with equality if and only if M is Fuchsian.

The principal result of [Glo15], which is the isolation results on $\delta(S_n, S'_n)$ when both surfaces move in $\text{Teich}(S)$ gives the counterpart of a theorem due to A. Sanders [San14] on the isolation of critical exponent for quasi-Fuchsian manifolds.

And the examples, are the counterpart of the examples given by C. McMullen in [McM99].

In a recent work with D. Monclair, we introduce a notion of Lorentzian Hausdorff dimension and show that the critical exponent is equal to the Hausdorff dimension of the limit set.

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