Critical exponent and Hausdorff dimension for quasi-Fuchsian AdS manifolds

Olivier Glorieux* And Daniel Monclair †

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Abstract

The aim of this article is to understand the geometry of limit sets in Anti-de Sitter space. We focus on a particular type of subgroups of $\mathrm{SO}(2,n)$ called quasi-Fuchsian groups (which are holonomies of globally hyperbolic manifolds). We define a Lorentzian analogue of critical exponent and Hausdorff dimension of the limit set. We show that they are equal and bounded from above by the topological dimension of the limit set. We also prove a rigidity result in dimension 3, which can be understood as a Lorentzian version of a famous Theorem of R. Bowen in the Hyperbolic case.

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†D. Monclair is supported by National Research Fund, Luxembourg

1 Introduction

Given Γ a uniform lattice of $\mathrm{SO}(1,n)$, we wish to understand its deformations in Lie groups containing $\mathrm{SO}(1,n)$. Deformations of Γ inside $\mathrm{SO}(1,n)$ are well understood : they are parametrized by the Teichmüller space for $n=2$ and they are trivial for $n \geq 3$ by Mostow rigidity. We can consider three naturals Lie groups which strictly contains $\mathrm{SO}(1,n)$: $\mathrm{SL}(n+1)$, $\mathrm{SO}(1,n+1)$, and $\mathrm{SO}(2,n)$. The three groups corresponds to three different geometries : deformations in $\mathrm{SL}(n+1)$ are associated to Hilbert geometry, deformations in $\mathrm{SO}(1,n+1)$ are (higher dimensional) quasi-Fuchsian groups, and finally deformations in $\mathrm{SO}(2,n)$ are associated to Anti-de Sitter geometry.

A way to understand how far these groups are from the original lattice in $\mathrm{SO}(1,n)$ is to look at some conjugacy invariants associated to it. Let us distinguish two types of invariants, dynamical and geometric. One of the classic dynamical invariants is the critical exponent which measures the growth of orbits. The Hausdorff dimension of the limit set provides a meaningful geometric invariant.

For deformations in $\mathrm{SO}(1,n+1)$ these two invariants coincide and it is known that this number is greater than $n-1$ with equality if and only if the deformation is trivial [Bow79, Pan89, Yue96, Bou96]. For deformations in $\mathrm{SL}(n+1)$, the known invariant is the critical exponent, and we have a similar rigidity result with reverse inequality [Cra09]. In this paper, we define and study the two invariants for deformations in $\mathrm{SO}(2,n)$.

1.1 Quasi-Fuchsian AdS groups

Let us give a brief description of the groups under study in this paper. More details are given in section 2.

The Anti-de Sitter space AdS^{n+1} is $\{q=-1\}$, where $q = -du^2 - dv^2 + dx_1^2 + \cdots + dx_n^2$ is the standard $(2,n)$ signature quadratic form on $\mathbb{R}^{n+2}$, endowed with the restrictions of q to tangent spaces.

As in any Lorentzian manifold, geodesics of AdS^{n+1} are classified in three different types: spacelike geodesics (for which tangent vectors are positive), timelike geodesics (negative tangent vectors) and lightlike geodesics (null tangent vectors).

In this paper, we will mostly study spacelike geodesics, which behave more like geodesics in Hyperbolic geometry.

Just as the hyperbolic space $\mathbb{H}^{n+1}$ can be compactified by a sphere at infinity realized as the projectivization of the isotropic cone of $\mathbb{R}^{n,1}$, we can compactify AdS^{n+1} by adding the projectivized isotropic cone of $\mathbb{R}^{n,2}$. We will denote this boundary by $\partial\mathrm{AdS}^{n+1}$.

A subset $\Lambda \subset \partial\mathrm{AdS}^{n+1}$ is called **acausal** if any two different points of Λ can be joined by a spacelike geodesic in AdS^{n+1}.

A group $\Gamma \subset \mathrm{SO}(2,n)$ is called **quasi-Fuchsian** if it preserves an acausal

subset $\Lambda \subset \partial\mathrm{AdS}^{n+1}$ which is a topological $(n-1)$-sphere, and such that the action of Γ on the convex hull of Λ is cocompact.
These groups are the natural equivalent of quasi-Fuchsian groups in hyperbolic geometry.

1.2 Critical exponent

We recall the classic definition of the critical exponent in metric spaces. Let G be a countable group acting on a metric space (X, d), and $o \in X$. The **critical exponent** $\delta(G, X)$ is

$$\delta(G,X) := \limsup_{R\to\infty} \frac{1}{R} \log \mathrm{Card}\{g \in G \,|\, d(go, o) \leq R\}.$$

This invariant is independent of $o \in X$ thanks to the triangle inequality. It measures the exponential growth rate of the orbit of G in X.
For example, by a simple argument of volume, we can see that the critical exponent of a uniform lattice of $\mathrm{SO}(n, 1)$ acting on $\mathbb{H}^n$ is equal to $n-1$. More generally this applies to fundamental group of compact Riemannian manifolds of negative curvature, where the critical exponent is equal to the exponential growth rate of the volume of balls. For a more thorough treatment we refer to the text of M. Peigné [Pei13] and F. Paulin [Pau97]
A famous theorem of R. Bowen [Bow79] in dimension 3 and Yue [Yue96] in higher dimension shows that the critical exponent of a quasi-Fuchsian representation in $\mathrm{SO}(1, n+1)$ is greater than $n-1$ with equality if and only if the deformation is trivial, that is the group preserves a totally geodesic copy of $\mathbb{H}^n$.

The main problem to define this invariant in AdS setting is that the AdS is not a metric space: there is no $\mathrm{SO}(2, n)$ invariant distance on AdS.
Our primary work will be to define a good notion of distance on the convex hull of a quasi-Fuchsian group, to define AdS critical exponent. This will be done in section 3.1. If two points are on the same space-like geodesic then their distance is defined to be the length on this geodesic (since its space-like, the induced metric on this geodesic is Riemannian), else their distance is defined to be 0. We call $d_{\mathrm{AdS}}(\cdot,\cdot)$ this function on $C(\Lambda) \times C(\Lambda)$ where $C(\Lambda)$ is the convex hull of the limit set.
With this definition of distance we then define the **critical exponent for quasi-Fuchsian groups** by

$$\delta_{\mathrm{AdS}}(\Gamma) := \limsup_{R\to\infty} \frac{1}{R} \log \mathrm{Card}\{\gamma \in \Gamma \,|\, d_{\mathrm{AdS}}(\gamma o, o) \leq R\},$$

we will see that this definition does not depend on the choice of the base point $o \in C(\Lambda)$.

1.3 Hausdorff dimension

The other invariant we want to generalize is the Hausdorff dimension of the limit set. In the hyperbolic case, it is known since the work of S. J. Patterson and D. Sullivan [Pat76, Sul79] that the Hausdorff dimension of the limit set of a quasi-Fuchsian group is equal to the critical exponent. It provides a link between the action of the group inside the Hyperbolic space and the fractal geometry of the limit set.
An acausal sphere of $\partial\mathrm{AdS}$ can be seen as a graph of a Lipschitz function, [MB12, Corollary 2.9], therefore its usual Hausdorff dimension is equal to $n-1$. Using the Lorenzian geometry of the boundary (which makes it isometric to the de Sitter space dS) we define a new notion of Hausdorff dimension, that we will denote by $\mathrm{Hdim}_{\mathrm{dS}}$, this will be done in Section 5. We obtain our first main result

Theorem 1.1. *Let $\Gamma \subset SO(2,n)$ be a quasi-Fuchsian group. Then*

$$\delta_{\mathrm{AdS}}(\Gamma) = \mathrm{Hdim}_{\mathrm{dS}}(\Lambda_\Gamma).$$

As a corollary of this theorem we find the upper bound on the critical exponent namely $\delta_{\mathrm{AdS}}(\Gamma) \leq n-1$.

1.4 Patterson-Sullivan densities

The main tool in the proof is the construction of a conformal density, which is a family of measures supported on the limit set indexed by points of the convex hull.

Definition 1.2. *A conformal density of dimension δ is a family of measures $(\nu_x)_{x\in C(\Lambda)}$ satisfying the following conditions:*

1. $\forall g \in \Gamma$, $g^*\nu_x = \nu_{gx}$ *(where $g^*\nu(E) = \nu(g^{-1}E)$)*
2. $\forall x, y \in D$, $\frac{d\nu_x}{d\nu_y}(\xi) = e^{-\delta\beta_\xi(x,y)}$
3. $\mathrm{supp}\,(\nu_x) = \Lambda$

We adapt a classical construction due to S.J. Patterson and D. Sullivan [Pat76, Sul79] in the hyperbolic case and obtain a conformal density of dimension $\delta_{\mathrm{AdS}}(\Gamma)$ in section 4. For a nice introduction of this theory we refer to the lecture notes of J.-F. Quint [Qui06].
One of the important steps in order to identify $\delta_{\mathrm{AdS}}(\Gamma)$ with the Hausdorff dimension of Λ is that the measure of a Lorentzian ball of radius r behaves like $r^{\delta_{\mathrm{AdS}}(\Gamma)}$ as $r \to 0$. This is a consequence of a result known as the Shadow Lemma (Theorem 4.5), proved in the hyperbolic case by Sullivan [Sul79].

Theorem 1.3. *Let ν be the Patterson-Sullivan density, and let $x \in C$. There is $c > 0$ such that for all $\xi \in \Lambda$, $r \in (0,1)$, we have:*

$$\frac{\nu_x(B_x(\xi, r))}{r^{\delta_\Gamma}} \in \left[\frac{1}{c}, c\right],$$

where $B_x(\xi, r)$ is the Lorentzian ball on the boundary.

Thanks to the shadow lemma we also prove that the Patterson-Sullivan measure is ergodic. We construct in the same manner a measure on the non-wandering set of the unit tangent for the geodesic flow a finite, invariant, ergodic measure: the Bowen-Margulis measure.

1.5 Rigidity

The second main result of this paper is the rigidity result that we obtain in dimension 3.

Theorem 1.4. *Let $\Gamma \subset \mathrm{SO}(2,2)$ be a quasi-Fuchsian group :*

$$\delta_{\mathrm{AdS}}(\Gamma) \leq 1,$$

with equality if and only if Γ preserves a totally geodesic $\mathbb{H}^2$.

It is the analogue of R. Bowen's Theorem [Bow79]. The proof of the inequality mimics a classic method using comparison of the volume of large balls, that we can found in [Kni95, Glo15b]. The main argument is to compare two distances on the boundary of the convex core. This two distances are the intrinsic and extrinsic distances coming from the AdS distance. We prove a reverse inequality compared to the Riemannian case: the extrinsic distance is greater than the intrinsic distance (up to a fixed additive constant). This inequality on distances has to be proven only on 2-dimensional Anti de Sitter space, where it follows by a simple computation.

The proof of the rigidity case uses the comparison of the two Patterson-Sullivan measures coming from the intrinsic and extrinsic distances. We show that if there is equality between the critical exponent of the intrinsic and extrinsic metric then the measures should be equivalent. By a classical remark using Bowen-Margulis measures, we show that the marked length spectrum of the boundary of the convex core and of the ambient, quasi-Fuchsian AdS^3-manifold are equal. This ends the proof thanks to a easy argument of two-dimensional hyperbolic geometry.

Remark: This result happens to be equivalent to a theorem of Bishop-Steger [BS91] for the action of surface group acting on $\mathbb{H}^2 \times \mathbb{H}^2$. Indeed the critical exponent is equal to the growth rate of periodic geodesics a quasi-Fuchsian AdS^3-manifolds. It is shown in [Glo15a] that this number is equal to the invariant studied by C. Bishop and T. Steger. The proof we propose is totally independent, and the version of this theorem in the paper (Theorem 6.5) is slightly stronger than the result of Bishop-Steger.

1.6 Plan of the paper

We start by recalling some aspect of AdS geometry. We emphasize on the differences and similarities between Anti-de Sitter space and the Hyperbolic space.

It is followed by a collection of geometric facts about asymptotic geometry for the distance d_{AdS}. We review the usual concepts of shadows, radial convergence, Busemann functions, and Gromov distances in our setting.

Section 4 is devoted to the construction and study of conformal densities.

In section 5, we define the Lorentzian Hausdorff dimension and prove Theorem 1.1.

Finally, the last section is devoted to the proof of Theorem 1.4.

2 AdS geometry

In this section we introduce the Anti-de Sitter space AdS^n and its properties. A nice introduction to Anti-de Sitter geometry can be found in [BBZ07] for AdS^3, and in [MB12] for arbitrary dimension.

2.1 Models for AdS and its boundary

As stated in the introduction, the Anti de Sitter space AdS^{n+1} is $q_{2,n}^{-1}(\{-1\})$, where $q_{2,n} = -du^2 - dv^2 + dx_1^2 + \cdots + dx_n^2$ is the standard $(2, n)$ signature quadratic form on $\mathbb{R}^{n+2}$, endowed with the restrictions of q to tangent spaces. We denote by $\langle \cdot \mid \cdot \rangle$ the corresponding bilinear form.
Since we want to compactify the Anti-de Sitter space, we will work with the (double cover of the) projective model. We view $\mathbb{S}^{n+1}$ as the quotient of $\mathbb{R}^{n+2} \setminus \{0\}$ by positive homotheties. Note that the associated quotient map $\mathbb{R}^{n+2} \setminus \{0\} \to \mathbb{S}^{n+1}$ is injective when restricted to AdS^{n+1} (which would not have been the case if we had chosen to work with the projective space instead of the sphere).

We define $\partial\mathrm{AdS}^{n+1}$ to be the quotient by positive homotheties of the isotropic cone $q_{2,n}^{-1}(\{0\}) \setminus \{0\}$.

This is a conformal Lorentzian manifold, called the Einstein Universe Ein^n (for some authors, this is a double cover of the Einstein Universe).

To define a topology on $\overline{\mathrm{AdS}^{n+1}} = \mathrm{AdS}^{n+1} \cup \partial\mathrm{AdS}^{n+1}$, we can embed both spaces in $\mathbb{S}^{n+1}$ through the quotient map defined above. This topology makes $\overline{\mathrm{AdS}^{n+1}}$ compact. A sequence (x_i) in AdS^{n+1} converges to $\xi \in \partial\mathrm{AdS}^{n+1}$ if and only if there is a sequence λ_i of positive numbers and a point ξ_0 on the half line ξ such that $\lambda_i x_i \to \xi_0$ in $\mathbb{R}^{n+2}$.

The AdS^3 case When $n = 2$ there is another convenient model, as AdS^3 is isometric to $\mathrm{SL}(2, \mathbb{R})$ endowed with a bi-invariant metric (its Killing form). Indeed we can see $\mathbb{R}^{2,2}$ as $M(2, \mathbb{R})$ endowed by the quadratic form $q = -\det$.

Then we can see $\mathrm{SL}(2,\mathbb{R})$ as the level $\{q=-1\}$ endowed with the restriction of q to tangent spaces. An isometry between AdS^3 and $\mathrm{SL}_2(\mathbb{R})$ is given by

$$\left|\begin{array}{ccc} \mathbb{R}^{2,2} & \longrightarrow & M(2,\mathbb{R}) \\ (x_1,x_2,x_3,x_4) & \longmapsto & \begin{pmatrix} x_1-x_3 & -x_2+x_4 \\ x_2+x_4 & x_1+x_3 \end{pmatrix} \end{array}\right. .$$

Inner product for points of $\overline{\mathrm{AdS}^{n+1}}$ Technically, a point $\xi\in\partial\mathrm{AdS}^{n+1}$ is a half line in $\mathbb{R}^{n+2}$, however, given two points $\xi,\eta\in\partial\mathrm{AdS}^{n+1}$ (resp. $x\in\partial\mathrm{AdS}^{n+1}$, $x\in\mathrm{AdS}^{n+1}$) we will use the notation $\langle\xi\,|\,\eta\rangle$ (resp. $\langle\xi\,|\,x\rangle$) as we will only be interested in the sign of this number, which does not depend on the choice of a point in the half line ξ.

With this convention, a point $\xi\in\partial\mathrm{AdS}^{n+1}$ satisfies $\langle\xi\,|\,\xi\rangle=0$.

We will also identify points of $\partial\mathrm{AdS}^{n+1}$ with points in $\mathbb{R}^{n+2}$ whenever the formula does not depend on the choice of a point in the half line (e.g. given $x\in\mathrm{AdS}^{n+1}$ and $\xi\in\partial\mathrm{AdS}^{n+1}$ such that $\langle x\,|\,\xi\rangle\neq 0$, the point $\frac{1}{\langle x\,|\,\xi\rangle}\xi\in q_{2,n}^{-1}(\{0\})$ does not depend on the choice of a point in the half line ξ).

2.2 Isometries of AdS^{n+1}

The group of orientation preserving isometries of AdS^{n+1} is the group $\mathrm{SO}(2,n)$ of linear transformations of $\mathbb{R}^{n+2}$ preserving the quadratic form $q_{2,n}$. It acts transitively on AdS^{n+1}.

The stabilizer of a point $x\in\mathrm{AdS}^{n+1}$ in $\mathrm{SO}(2,n)$ is isomorphic to $\mathrm{SO}(1,n)$. For $x_0=(1,0,\dots,0)$, the associated inclusion $\mathrm{SO}(1,n)\subset\mathrm{SO}(2,n)$ corresponds to the standard inclusion by block-diagonal matrices, so AdS^{n+1} can be seen as the homogeneous space $\mathrm{SO}(2,n)/\mathrm{SO}(1,n)$.

2.3 Geodesics

Definition 2.1. *Geodesics are intersections of* 2 *dimensional plane in* $\mathbb{R}^{n+2}$ *with* AdS^{n+1}.

This definition is equivalent to the classical notion of geodesics in pseudo-Riemannian geometry, however this is the characterisation that will be of interest to us.

Given two points $x,y\in\mathrm{AdS}^{n+1}$, there is a unique geodesic of AdS^{n+1} joining x and y (the intersection of AdS^{n+1} and the 2-plane in $\mathbb{R}^{n+2}$ generated by x and y).

As in any Lorentzian manifold, geodesics of AdS^{n+1} are classified in three different types: spacelike geodesics (for which tangent vectors are positive for $q_{2,n}$), timelike geodesics (negative tangent vectors) and lightlike geodesics (null tangent vectors).

The type of the geodesic joining x and y can be linked to the inner product. It is:

- Spacelike if and only if $\langle x \,|\, y\rangle < -1$,
- Lightlike if and only if $\langle x \,|\, y\rangle = -1$,
- Timelike if and only if $\langle x \,|\, y\rangle \in (-1, 1]$.

Note that in particular two different points are not necessarily joined by a geodesic.

Given two distinct points $x, y \in \mathrm{AdS}^{n+1}$, we will denote by (xy) the geodesic passing through x and y.

The AdS^3 **case** In the $\mathrm{SL}(2,\mathbb{R})$ model, the geodesics passing through Id are exactly the 1-parameter subgroups. They are conjugate to the classical 1−parameter subgroups of $\mathrm{SL}(2,\mathbb{R})$ of Cartan decomposition.

- Space-like geodesics are conjugated to $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$
- Light-like geodesics are conjugated to $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$
- Time-like geodesics are conjugated to $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$

In other words, spacelike geodesics passing through Id are hyperbolic 1-parameter groups, timelike geodesics passing through Id are elliptic 1-parameter groups, and spacelike geodesics passing through Id are parabolic 1-parameter groups.

The inner product can also be interpreted in terms of traces: if $A, B \in \mathrm{SL}(2,\mathbb{R})$, then $\langle A \,|\, B\rangle = -\frac{\mathrm{Tr}(A^{-1}B)}{2}$, which shows for instance that the geodesic joining Id and A is spacelike if and only if $\mathrm{Tr}(A) > 2$, which means that the 1-parameter group generated by A is hyperbolic, and leads to the same description of the types of geodesics in terms of 1-parameter groups.

Geodesics and $\partial\mathrm{AdS}^{n+1}$**.** Not all geodesics of AdS^{n+1} have endpoints on $\partial\mathrm{AdS}^{n+1}$. Conversely, not all pairs of points of $\partial\mathrm{AdS}^{n+1}$ can be joined by a geodesic in AdS^{n+1}. This is a major difference with hyperbolic geometry. However, the situation is nicer if we restrict ourselves to spacelike geodesics. Indeed, timelike geodesics are closed, so they never meet the boundary, and lightlike geodesics meet the boundary at exactly one point.

Given $x \in \mathrm{AdS}^{n+1}$ and $\xi \in \partial\mathrm{AdS}^{n+1}$, there is a spacelike geodesic passing though x with endpoint ξ if and only if $\langle x \,|\, \xi \rangle < 0$. In this case, we can find a explicit paramatrization of this geodesic:

$$\begin{aligned} f(s) &= \cosh(s)x + \sinh(s)\left(\frac{\xi}{|\langle x \,|\, \xi\rangle|} - x\right) \\ &= e^{-s}x + \frac{\sinh s}{|\langle x \,|\, \xi\rangle|}\xi. \end{aligned}$$

We will denote by $(x\xi)$ this geodesic, and by $[x\xi)$ the half geodesic going from x to ξ.

Given two points $\xi, \eta \in \partial\mathrm{AdS}^{n+1}$, there is a spacelike geodesic with endpoints ξ and η if and only if $\langle \xi \,|\, \eta \rangle < 0$.

2.4 Affine domains

One of the interesting properties of AdS^{n+1} is the duality between points and hyperplanes.

Definition 2.2. *Let $x \in \mathrm{AdS}$. Its dual hyperplane is the set*

$$x^* := \{y \in \mathrm{AdS} \mid \langle x \,|\, y\rangle = 0\}.$$

The dual hyperplane x^* is a totally geodesic embedded copy of $\mathbb{H}^n$ in AdS^{n+1}. Conversely, any totally geodesic embedded copy of $\mathbb{H}^n$ in AdS^{n+1} is equal to x^* for a unique point $x \in \mathrm{AdS}^{n+1}$.

Definition 2.3. *Let $x \in \mathrm{AdS}^{n+1}$. The affine domain associated to x is*

$$U(x) = \{y \in \mathrm{AdS}^{n+1} \mid \langle x \,|\, y\rangle < 0\}.$$

The de Sitter domain associated to x is

$$\partial U(x) = \{\xi \in \partial\mathrm{AdS}^{n+1} \mid \langle x \,|\, \xi\rangle < 0\}.$$

In order to understand why $U(x)$ is called an affine domain, consider $x_0 = (1, 0, \dots, 0) \in \mathbb{R}^{n+2}$.

The affine domain $U(x_0)$ consists of points $(u, v, x_1, \dots, x_n) \in \mathrm{AdS}^{n+1}$ such that $u > 0$. The map $(u, v, x_1, \dots, x_n) \mapsto (\frac{v}{u}, \frac{x_1}{u}, \dots, \frac{x_n}{u})$ maps $U(x_0)$ to an open set V of $\mathbb{R}^{n+1}$, and sends geodesics to affine lines in $\mathbb{R}^{n+1}$.

More precisely, if we denote by $q_{1,n}$ the standard quadratic form of signature $(1, n)$ on $\mathbb{R}^{n+1}$, then $V = q_{1,n}^{-1}((-\infty, 1))$ is the interior of the hyperboloid $H = q_{1,n}^{-1}(\{1\})$, which is the image of $\partial U(x_0)$ through the same map.

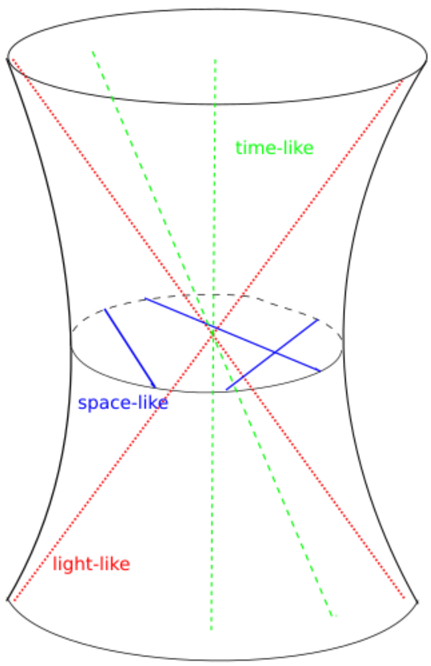

Figure 1: Geodesics of AdS^3 in an affine domain.

In this model, it is easy to understand why a pair of points of $\partial\mathrm{AdS}^{n+1}$ is not always joined by a geodesic in AdS^{n+1}, this is because the interior of the hyperboloid H is not convex (see Figure 2.4).

Note that a similar description of $U(x)$ and $\partial U(x)$ is valid for any point $x \in \mathrm{AdS}^{n+1}$ (because the isometry group $\mathrm{SO}(2,n)$ acts transitively on AdS^{n+1}).

2.5 The geometry of $\partial\mathrm{AdS}^{n+1}$

Given $x \in \mathrm{AdS}^{n+1}$, the de Sitter domain $\partial U(x) = \{\xi \in \partial\mathrm{AdS}^{n+1} \mid \langle x \mid \xi\rangle < 0\}$ can be equipped with a canonical Lorentzian metric g_x such that $(\partial U(x), g_x)$ is isometric to the de Sitter space dS^n.

The de Sitter space is the Lorentzian analogue of the sphere (i.e. the standard Lorentzian manifold of constant curvature 1). To define it, consider the quadratic form $\langle\cdot\mid\cdot\rangle_{1,n}$ of signature $(1,n)$ on $\mathbb{R}^{n+1}$, the de Sitter space dS^n is the level $\{v \in \mathbb{R}^{n+1} | \langle v \mid v\rangle_{1,n} = 1\}$ equipped with the restriction of $\langle\cdot\mid\cdot\rangle_{1,n}$ to its tangent spaces. Its isometry group is $\mathrm{SO}(1,n)$.

Considering the point $x_0 = (1,0,\ldots,0) \in \mathrm{AdS}^{n+1}$, we find a diffeomorphism $\partial U(x_0) \to \mathrm{dS}^n$ by sending the half line generated by $(\xi_0,\ldots,\xi_{n+1})$ to $(\frac{\xi_1}{\xi_0},\ldots,\frac{\xi_{n+1}}{\xi_0})$. Pulling back the metric of dS^n to $\partial U(x_0)$ yields a metric g_{x_0}

whose isometry group is the stabilizer of x_0 in $\mathrm{SO}(2,n)$.

Given any other point $x \in \mathrm{AdS}^{n+1}$, consider $\gamma \in \mathrm{SO}(2,n)$ such that $\gamma.x = x_0$. We then have $\gamma.\partial U(x) = \partial U(x_0)$, and we can define a Lorentzian metric g_x on $\partial U(x)$ as the pullback $\gamma^* g_{x_0}$. It does not depend on the choice of the element $\gamma \in \mathrm{SO}(2,n)$ because the metric g_{x_0} is invariant under the stabilizer of x_0.

Note that the isometry group of $(\partial U(x), g_x)$ is exactly $\mathrm{Stab}(x)$.

It is not possible to find a Lorentzian metric defined on the whole boundary $\partial\mathrm{AdS}^{n+1}$ which is invariant under $\mathrm{SO}(2,n)$. However, there is an invariant Lorentzian conformal structure. Indeed, the metrics defined on the de Sitter domains define the same conformal class on their intersection. This conformal Lorentzian manifold is called the Einstein Universe Ein^n (or a double cover of the Einstein Universe for some authors), it is conformally diffeomorphic to $(\mathbb{S}^1 \times \mathbb{S}^{n-1}, -d\theta^2 + g_{\mathbb{S}^{n-1}})$ where $d\theta^2$ is the Riemannian metric of length 2π on $\mathbb{S}^1$ and $g_{\mathbb{S}^{n-1}}$ is the spherical metric on $\mathbb{S}^{n-1}$.

2.6 Acausal sets, convex hulls and black domains

Definition 2.4. *A subset $\Lambda \subset \partial\mathrm{AdS}^{n+1}$ is acausal if $\langle \xi \mid \eta \rangle < 0$ for any two distinct points $\xi, \eta \in \Lambda$.*

This means that a set is acausal if any two points can be joined by a spacelike geodesic in AdS^{n+1}.
Note that an acausal set Λ is always included in the boundary of an affine domain: choose two distinct points $\xi, \eta \in \Lambda$, and let x be any point on the geodesic $(\xi\eta)$. A simple computation shows that $\Lambda \subset \partial U(x)$. This allows us to define the convex hull of Λ.

Definition 2.5. *Let $\Lambda \subset \partial\mathrm{AdS}^{n+1}$ be an acausal set. We denote by $\overline{C(\Lambda)} \subset \overline{\mathrm{AdS}^{n+1}}$ the convex hull of Λ, computed in any affine domain containg Λ (it does not depend on the choice of an affine domain).*
We also let $C(\Lambda) = \overline{C(\Lambda)} \cap \mathrm{AdS}^{n+1} = \overline{C(\Lambda)} \setminus \Lambda$.

Another important subset of AdS^{n+1} associated to Λ is its black domain (or invisible domain).

Definition 2.6. *Let $\Lambda \subset \partial\mathrm{AdS}^{n+1}$ be an acausal set. Its black domain is $E(\Lambda) = \{x \in \mathrm{AdS}^{n+1} \mid \langle x \mid \xi \rangle < 0 \ \forall \xi \in \Lambda\}$*

One can check that $E(\Lambda)$ is convex, and that it contains $C(\Lambda)$.

Lemma 2.7. *If $x \in E(\Lambda)$, then the dual hyperplane x^* is disjoint from $C(\Lambda)$.*

Proof. It comes from the definition of the black domain $E(\Lambda)$ that x^* is disjoint from Λ. The connected component of the complement of $\overline{x^*}$ in $\overline{\mathrm{AdS}^{n+1}}$ that contains Λ is a convex set, so it must contain $\overline{C(\Lambda)}$, hence the result. □

2.7 AdS quasi-Fuchsian groups

Definition 2.8. *A group $\Gamma \subset \mathrm{SO}(2,n)$ is called quasi-Fuchsian if it is discrete, torsion-free, preserves a acausal set Λ which is homeomorphic to the $n-1$ sphere, and the action of Γ on $C(\Lambda)$ is cocompact.*

Remark: This definition is slightly different from the definition of [MB12]. Indeed, they ask that a quasi-Fuchsian group must be isomorphic to a uniform lattice in $\mathrm{SO}(1,n)$, which is not necessary for our work.

The acausal set Λ involved in the definition of a quasi-Fuchsian group Γ, called the limit set, is unique because it is the closure of the set of attractive fixed points on $\partial\mathrm{AdS}^{n+1}$ of elements of Γ.

Proposition 2.9 ([MB12])**.** *If $\Gamma \subset \mathrm{SO}(2,n)$ is quasi-Fuchsian, then Γ is Gromov-hyperbolic, and the action of Γ on Λ is topologically conjugate to the action on its Gromov boundary $\partial_\infty\Gamma$.*

In particular, the action of Γ on the set of triples of distinct points is proper and cocompact, which will be of some use to us.

Elements of Γ have a north-south dynamic as any torsion free hyperbolic group acting on its boundary. It is a consequence of [MB12, Proposition 6.6].

Proposition 2.10. *If $\Gamma \subset \mathrm{SO}(2,n)$ is a quasi-Fuchsian group, then every element $\gamma \in \Gamma \setminus \{\mathrm{Id}\}$ acts on Λ with exactly two fixed points: $\gamma^\pm$. For every $\xi \in \Lambda \setminus \{\gamma^\pm\}$, we have $\lim_{n\to\pm\infty} \gamma^n\xi = \gamma^\pm$.*

Definition 2.11. *For every element $\gamma \in \Gamma \setminus \{\mathrm{Id}\}$ we call the spacelike geodesic $(\gamma^- g^+)$ the axis of γ.*

Remark: In the language of Anosov representations [Lab06], the acausality condition is equivalent to the transversality of the limit maps. It is also a natural requirement in Lorentzian geometry as it guarantees that the quotient $E(\Lambda)/\Gamma$ is globally hyperbolic (see subsection 2.9).

2.8 AdS Fuchsian groups and their deformations

The standard examples of quasi-Fuchsian groups are Fuchsian groups. Given a point $x \in \mathrm{AdS}^{n+1}$, we have an injection $i_x : \mathrm{SO}(1,n) \hookrightarrow \mathrm{SO}(2,n)$ given by identifying $\mathrm{SO}(1,n)$ and the stabilizer of x in $\mathrm{SO}(2,n)$. Given a torsion free uniform lattice $\Gamma_0 \subset \mathrm{SO}(1,n)$, we consider $\Gamma = i_x(\Gamma_0)$. Such a group is called Fuchsian, and it is the simplest example of a quasi-Fuchsian group. Indeed, the group Γ preserves $\partial x^* \subset \partial \mathrm{AdS}^{n+1}$ which is an acausal sphere. The convex hull $C(\partial x^*) = x^*$ is a totally geodesic copy of $\mathbb{H}^n$ in AdS^{n+1}, and the action of Γ on $C(x^*)$ is conjugate to the action of Γ_0 on $\mathbb{H}^n$, which is cocompact.

Note that a quasi-Fuchsian group $\Gamma \subset \mathrm{SO}(2,n)$ is Fuchsian if and only if it preserves a totally geodesic copy of $\mathbb{H}^n$. Indeed, if $P \subset \mathrm{AdS}^{n+1}$ is such a hyperplane, there is a unique point $x \in \mathrm{AdS}^{n+1}$ such that $P = x^*$, and $\Gamma \subset \mathrm{Stab}(x)$, i.e. $\Gamma = i_x(\Gamma_0)$ for a certain group $\Gamma_0 \subset \mathbb{H}^n$. Furthermore, one necessarily has $\Lambda = \partial P$ (because Λ is the closure of attractive fixed points of elements of Γ, and those fixed points lie on ∂x^*), hence $C(\Lambda) = P$. The action of Γ on $C(\Lambda)$ being cocompact, we see that Γ_0 is a uniform lattice in $\mathrm{SO}(1,n)$.

One can also consider deformations of Fuchsian groups, i.e. groups $\Gamma = \rho_1(\Gamma_0)$ where $\Gamma_0 \subset \mathrm{SO}(1,n)$ is a uniform lattice and $(\rho_t)_{t\in[0,1]}$ is a continuous path of representations with $\rho_0 = i_x$ for some $x \in \mathrm{AdS}^{n+1}$.

Theorem 2.12. *[Bar15] Deformations of Fuchsian groups are quasi-Fuchsian.*

Non trivial deformations exist in any dimension, the standard construction being the so called bending deformations [JM87].

2.9 Spatially compact AdS manifolds

Quasi-Fuchsian groups of $\mathrm{SO}(2,n)$ correspond to holonomy representations of geometric structures on some Lorentzian manifolds.

If $\Gamma \subset \mathrm{SO}(2,n)$ is quasi-Fuchsian, then Γ acts properly discontinuously and freely on the black domain $E(\Lambda)$ (furthermore, the black domain is the largest open set of AdS^{n+1} on which the action is properly discontinuous) [Mes07, B+08].

The quotient $M = E(\Lambda)/\Gamma$ is a Lorentzian manifold of dimension $n+1$ locally isometric to AdS^{n+1}, so it has constant sectional curvature -1. One of the important properties of M is that it is globally hyperbolic: there is a hypersurface $\Sigma \subset M$ that meets every inextensible causal curve exactly once (a causal curve is a curve $c : I \subset \mathbb{R} \to M$ such that $g_{c(t)}(\dot{c}(t), \dot{c}(t)) \leq 0$ and $\dot{c}(t) \neq 0$ for all $t \in I$, it is inextensible if it is not the reparametrization of a curve defined on a larger interval). Such a hypersurface is called a Cauchy hypersurface, an example is given by $\partial_+ C(\Lambda)/\Gamma$. Furthermore, it is spatially compact: Cauchy hypersurfaces are compact.

Conversely, given a spatially compact Lorentzian manifold M^{n+1} of constant curvature -1, one can consider its holonomy representation $\rho : \pi_1(M) \to \mathrm{SO}(2,n)$ (indeed, a Lorentzian manifold of constant curvature -1 possesses a (G,X)-structure, where $G = \mathrm{SO}(2,n)$ and $X = \mathrm{AdS}^{n+1}$). If $\pi_1(M)$ is Gromov hyperbolic, then $\Gamma = \rho(\pi_1(M))$ is quasi-Fuchsian. Assuming that M is maximal (i.e. cannot be embedded in a larger Lorentzian manifold with the same properties), the group Γ characterizes M up to isometry [Mes07, B+08].

One of the starting points in the study of spatially compact Lorentzian manifolds of curvature -1 is the following theorem of Mess, which gives a characterization of quasi-Fuchsian subgroups of $\mathrm{SO}(2,2)$ in terms of the exceptionnal isomorphism $\mathrm{SO}_\circ(2,2) \approx \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$.

Theorem 2.13 ([Mes07, B+08])**.** *A group $\Gamma \subset \mathrm{SO}_\circ(2,2) \approx \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ is quasi-Fuchsian if and only if there is a closed surface S of genus $g \geq 2$, and two hyperbolic metrics h_1, h_2 on S such that*

$$\Gamma = \{(\rho_1(\gamma), \rho_2(\gamma)) \,|\, \gamma \in \pi_1(S)\}$$

where $\rho_1, \rho_2 : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ are the holonomy representations of h_1, h_2.

This is a Lorentzian analogue of the Bers double uniformization theorem [Ber72]. In this description of quasi-Fuchsian subgroups of $\mathrm{SO}(2,2)$ (called the Mess parametrization), Fuchsian groups correspond to pairs (ρ_1, ρ_2) where ρ_1 and ρ_2 are conjugate in $\mathrm{PSL}(2,\mathbb{R})$ (i.e. they represent the same point in the Teichmüller space of S).

3 Geometric toolbox

This section contains all the geometric lemmas that will be used in the rest of the paper. We start by the definition of the Lorentzian distance in the convex hull, d. It is by definition semi-definite and we prove that it satisfies a triangle inequality up to fixed additive constant. We then study the geometry of the limit set of Γ from the point of view of the distance d, in particular we show that every limit points are radial limit points. Finally we define the Gromov product and distance on the boundary as in the case of Gromov hyperbolic spaces and study their properties.

3.1 Triangle inequality

Definition 3.1. *Given $x, y \in C(\Lambda)$, we denote by $d(x,y)$ the length of the geodesic of AdS^{n+1} joining x and y if this geodesic is spacelike, and 0 if it is causal.*

Proposition 3.2. *Let $x, y \in C(\Lambda)$. Then x and y are joined by a space like geodesic if and only if $\langle x \,|\, y\rangle < -1$, moreover in that case*

$$d(x, y) = \text{Argcosh}(|\langle x \,|\, y\rangle|).$$

Proof. Since the action of $\mathrm{SO}(2, n)$ on spacelike geodesics of AdS^n is transitive, and both sides of the equality are invariant by this action, it is sufficient to prove the formula for $x = (0, 1, 0, ..., 0)$ and $y = (0, \cosh(t), \sinh(t), 0, ..., 0)$. Since the curve $t \to (0, \cosh(t), \sinh(t), 0, ..., 0)$ is a space like geodesic of unit speed, the result is clear. □

Lemma 3.3. *The function $F : E(\Lambda)^2 \times C(\Lambda) \to \mathbb{R}$ defined by $F(x, y, z) = \frac{\langle x|y\rangle}{\langle x|z\rangle\langle z|y\rangle}$ extends to a continuous bounded function on $\overline{E(\Lambda)}^2 \times C(\Lambda)$.*

Proof. First, notice that F is well defined by Lemma 2.7 (x^* is defined by $\langle x|\cdot\rangle = 0$).
In order to extend F to $\overline{E(\Lambda)}^2 \times C(\Lambda)$, simply notice that the formula still makes sense if x or y is in $\overline{E(\Lambda)}$ (but not for $z \in \Lambda$). As it is defined by ratios of scalar products, it is continuous.
Let $K \subset C(\Lambda)$ be a compact set such that $\Gamma.K = C(\Lambda)$. Since F is Γ-invariant, its values are all taken on the compact set $\overline{E(\Lambda)}^2 \times K$ and F is continuous, which shows that F is bounded. □

Theorem 3.4 (Triangle inequality)**.** *There is a constant $k > 0$ such that $d(x, y) \leq d(x, z) + d(z, y) + k$ for all $x, y, z \in C(\Lambda)$.*

Proof. If x and y are causally related, the inegality is automatic because the left hand side is 0, so we can assume that x and y are separated by a spacelike geodesic. This means that $d(x, y) = \text{Argcosh}|\langle x|y\rangle|$.
First assume that x, y, z are pairwise joined by spacelike geodesics. Since we have $\ln t \leq \text{Argcosh} t \leq \ln t + \ln 2$ for all $t > 0$, in order to show that $(x, y, z) \mapsto d(x, y) - d(x, z) - d(z, y)$ is bounded from above, it is enough to show that it is the case for $(x, y, z) \mapsto \ln \frac{|\langle x|y\rangle|}{|\langle x|z\rangle\langle z|y\rangle|}$. This is true because of Lemma 3.3.
Now assume that $d(x, z) = d(y, z) = 0$. We wish to show that $d(x, y)$ is bounded. Up to the action of Γ, it is enough to show this for z in a compact fundamental domain K for the action on $C(\Lambda)$. The points x, y belong to the set $J(K) = \{p \in C(\Lambda) | \exists q \in K \; d(p, q) = 0\}$ which is compact. Indeed, if $p_i \in J(K)$ goes to infinity, there is a sequence $q_i \in K$ such that $d(p_i, q_i) = 0$. Up to a subsequence, we may assume thatand $q_i \to q \in K$ and $p_i \to \xi \in \Lambda$ (i.e. $\lambda_i p_i \to \xi_0$ in $\mathbb{R}^{n+2}$ for a sequence of positive numbers $\lambda_i \to 0$). By definition of $d(p_i, q_i) = 0$, we have $|\langle p_i \,|\, q_i\rangle| \leq 1$, hence $\langle \xi_0 \,|\, q\rangle = \lim\langle \lambda_i p_i \,|\, q_i\rangle = 0$, which is absurd because of Lemma 2.7. It follows that $d(x, y)$ for $x, y \in J(K)$ is bounded.
Finally assume that $d(x, z) > 0$ and $d(y, z) = 0$. Again, we can assume that

z is in a compact fundamental domain K, and since $y \in J(K)$ we see that y also lies in a compact set. We now have to show that $x \mapsto d(x, y) - d(x, z)$ is bounded on $C(\Lambda)$. This is true because it extends to a continuous function on $\overline{C(\Lambda)}$ (the Buseman function). $\square$

Definition 3.5. *We will call Lorentzian ball or simply ball centred at $x \in C(\Lambda)$ of radius R the set $B(x, R) := \{y \in C(\Lambda) \mid d(x, y) \leq R\}$.*

The previous Theorem shows that for all $x, y \in C(\Lambda)$ for all $R > 0$ we have $B(x, R) \subset B(y, R + d(x, y) + k)$.

Balls of radius 0 are cones and those of positive radius are hyperboloids (more precisely they are the intersection of cones or hyperboloids with $C(\Lambda)$.)

3.2 Limit set

Lemma 3.6. *Two space like rays with the same endpoint are at bounded distance.*

Proof. The equation of the geodesic joining x to η is:

$$x(t) = e^{-t}x - \frac{\sinh t}{\langle x|\eta\rangle}\eta$$

The equation of the geodesic joining y to η is:

$$y(t) = e^{-t}y - \frac{\sinh t}{\langle y|\eta\rangle}\eta$$

If $z \in [y\eta)$, we write $z = y(t)$ for some $t \geq 0$, and compute $d(z, x(t)) = \text{Argcosh}|\langle z|x(t)\rangle|$:

$$\begin{aligned}\langle z|x(t)\rangle &= \langle e^{-t}x - \frac{\sinh t}{\langle x|\eta\rangle}\eta | e^{-t}y - \frac{\sinh t}{\langle y|\eta\rangle}\eta\rangle \\ &= e^{-2t}\langle x|y\rangle - e^{-t}\sinh t\left(\frac{\langle x|\eta\rangle}{\langle y|\eta\rangle} + \frac{\langle x|\eta\rangle}{\langle y|\eta\rangle}\right)\end{aligned}$$

Since K and Λ are compact, there is a constant $\ell > 0$ such that $|\langle x|y\rangle| \leq \ell$ for all $x, y \in K$, and $\frac{\langle x|\eta\rangle}{\langle y|\eta\rangle} + \frac{\langle x|\eta\rangle}{\langle y|\eta\rangle} \leq \ell$ for all $(x, y, \eta) \in K^2 \times \Lambda$. We find that $|\langle z|x(t)\rangle| \leq 2\ell$, so by letting $d = \text{Argcosh}2\ell$ we get $d(z, [x\eta)) \leq d$. $\square$

We follow the usual definition of Buseman functions, in the hyperbolic case they are deeply studied in [Bal85]

Definition 3.7. *The Buseman function centred at ξ by $\forall x, y \in E(\Lambda)$ is:*

$$\beta_\xi(x, y) = \ln\left(\frac{\langle \xi \mid x\rangle}{\langle \xi \mid y\rangle}\right).$$

Lemma 3.8.

$$\lim_{z\to\xi} d(z,x) - d(z,y) = \beta_\xi(x,y).$$

Proof. Let $\xi \in \Lambda$, $|\langle \xi \,|\, x\rangle| \neq 0$ since $x \in D$. Hence, if $z \to \xi \in \Lambda$, then $|\langle z \,|\, x\rangle| \to \infty$. The same applies for $|\langle z \,|\, y\rangle|$. In particular implies that z is space like related to x and y, and that

$$d(z,x) - d(z,y) = \text{Argcosh}(|\langle z \,|\, x\rangle|) - \text{Argcosh}(|\langle z \,|\, y\rangle|)$$

Hence

$$\begin{aligned} \lim_{z\to\xi} d(z,x) - d(z,y) &= \lim_{z\to\xi} \ln\left(\frac{|\langle z \,|\, x\rangle|)}{|\langle z \,|\, y\rangle|}\right) \\ &= \ln\left(\frac{|\langle \xi \,|\, x\rangle|}{|\langle \xi \,|\, y\rangle|}\right) \\ &= \beta_\xi(x,y). \end{aligned}$$

□

3.3 Shadows

Definition 3.9 (Shadows)**.** *Let $x, y \in C(\Lambda)$, and $r > 0$. The shadow $\mathcal{S}_r(x,y)$ is $\{\xi \in \Lambda \,|\, [x,\xi) \cap B(y,r) \neq \emptyset\}$, where $B(y,r)$ is the Lorentzian ball.*

Remark: This is slightly different from the usual definition of shadows as we require that points in shadows lie on the limit set.

Lemma 3.10. *[Qui06, Lemma 4.1] Let $x, y \in C(\Lambda)$ and $r > 0$. For all $\xi \in \mathcal{S}_r(x,y)$, one has:*

$$d(x,y) - 2r - 2k \leq \beta_\xi(x,y) \leq d(x,y) + k.$$

Proof. Let $x(t)$ be the geodesic such that $x(0) = x$ and $x(+\infty) = \xi$. By definition of the shadow, there is $t_0 \geq 0$ such that $d(y, x(t_0)) < r$.
Since $x(+\infty) = \xi$, one has $\beta_\xi(x,y) = \lim_{t\to+\infty} d(x,x(t)) - d(y,x(t))$.

$$d(y,x(t)) \leq d(y,x(t_0)) + d(x(t_0),x(t)) + k$$

If $t \geq t_0$, one has $d(x(t_0),x(t)) = d(x,x(t)) - d(x,x(t_0))$, hence

$$d(y,x(t)) \leq r + d(x,x(t)) - d(x,x(t_0)) + k$$

$$d(x,x(t)) - d(y,x(t)) \geq d(x,x(t_0)) - r - k$$

However, $d(x,x(t_0)) \geq d(x,y) - d(y,x(t_0)) - k$, hence:

$$d(x, x(t)) - d(y, x(t)) \geq d(x, y) - 2(r + k)$$

Letting $t \to +\infty$ gives the left hand side of the desired inequality. For the right hand side, simply notice that $d(x, x(t)) - d(y, x(t)) \leq d(x, y) + k$. □

3.4 Radial convergence

Definition 3.11. *A limit point ξ is said radial, if there exists a sequence $g_i \in \Gamma$ and a point x such that $g_i x$ converges to ξ and is at bounded distance of one (any) space-like ray with endpoint ξ. In that case we say that the sequence $g_i x$ converges radially to ξ.*

Lemma 3.12. *If ξ is radial there exists a sequence $\gamma_i \in \Gamma$ and a constant $L > 0$ such that :*

$$|\beta_\xi(x, \gamma_i x) - d(x, \gamma_i x)| \leq L.$$

Proof. If ξ is radial, by definition there exists a sequence $\gamma_i \in \Gamma$, a point $x \in \mathrm{AdS}$ and a constant L such that the sequence $d(\gamma_i x, [x, \xi)) \leq L$. This means that $\xi \in \mathcal{S}_L(x, \gamma_i x)$. Conclude by Lemma 3.10. □

Lemma 3.13. *Let $\gamma \in \Gamma$ where Γ is a quasi-Fuchsian* AdS *group. Then γ^+ is radial.*

Proof. For all point x on the axis of γ, the sequence $\gamma^i x$ converges radially to γ^+. □

Lemma 3.14. *Every limit point of a quasi-Fuchsian* AdS *group is radial.*

Proof. This is due to the cocompactness of the action on the convex core. Let $x \in C(\Lambda)$ and $\xi \in \Lambda$. Let K be a compact fundamental domain for the action of Γ on $C(\Lambda)$. The geodesic ray $[x, \xi)$ is cover by infinitely many translates of K [MB12, Lemma 7.5], says $g_i K$. Then the sequence $g_i x$ converges radially to ξ. □

Lemma 3.15. *We choose an enumeration $\Gamma = \{\gamma_p : p \in \mathbb{N}\}$ of Γ.*

$$\Lambda = \bigcup_{r>0} \bigcap_{N \in \mathbb{N}} \bigcup_{p \geq N} \mathcal{S}_r(x, \gamma_p.x)$$

Proof. The fact that $\Lambda \supset \bigcup_{r>0} \bigcap_{N\in\mathbb{N}} \bigcup_{p\geq N} \mathcal{S}_r(x, \gamma_p.x)$ comes from the definition of shadows (they are subsets of Λ).
Let $\xi \in \Lambda$. We wish to find $r > 0$ such that $\xi \in \bigcap_{N\in\mathbb{N}} \bigcup_{p\geq N} \mathcal{S}_r(x, \gamma_p.x)$, i.e. such that there is an infinite number of elements $\gamma \in \Gamma$ such that $\xi \in \mathcal{S}_r(x, \gamma.x)$. For this, we choose a sequence $g_i \in \Gamma$ such that g_i converges radially to ξ. Let $r > 0$ be such that all $g_i.x$ are at distance at most r from the half geodesic $[x\xi)$. We then have $\xi \in \mathcal{S}_r(x, g_i.x)$ for all $i \in \mathbb{N}$, hence the result. □

3.5 Gromov distance

We denote by $\Lambda^{(2)}$ the pairs of distinct points of Λ.
The Gromov product of three points $x, y, z \in C(\Lambda)$ is:

$$(x|y)_z = \frac{1}{2}(d(x,z) + d(y,z) - d(x,y)).$$

It extends to a continuous function on $\Lambda^{(2)} \times C(\Lambda)$, and we have $\forall \xi, \eta \in \Lambda^{(2)}$, $\forall x \in C(\Lambda)$:

$$(\xi|\eta)_x = \frac{1}{2}\text{Log}\frac{-2\langle \xi|x\rangle\langle x|\eta\rangle}{\langle \xi|\eta\rangle}.$$

Note that $(\xi|\eta)_x = \frac{1}{2}(\beta_\xi(x,y) + \beta_\eta(x,y))$ for any $y \in (\xi\eta)$.
For $x \in C(\Lambda)$ and $\xi, \eta \in \Lambda^{(2)}$, we set $d_x(\xi,\eta) = e^{-(\xi|\eta)_x}$. The explicit formula is:

$$d_x(\xi,\eta) = \sqrt{\frac{-\langle \xi|\eta\rangle}{2\langle \xi|x\rangle\langle x|\eta\rangle}}$$

Note that we always have $d_x(\xi,\eta) \leq 1$, and $d_x(\xi,\eta) = 1$ if and only if $x \in (\xi\eta)$. Indeed, if $x \in C(\Lambda)$ and $\xi, \eta \in \Lambda$, then the affine subspace spanned by x, ξ, η is a totally geodesic copy of $\mathbb{H}^2$, so this follows from the fact that in $\mathbb{H}^2$, the distance d_x is the half of the chordal distance, when x is seen as the centre of the unit disk.

Remark that since for all y, z, x, x', we have $|(z,y)_x - (z,y)_{x'}| \leq 2d(x,x') + 2k$ the function d_x and $d_{x'}$ are bi-Lipschitz equivalent.

The function d_x is symmetric and $d_x(\xi,\eta) = 0 \iff \xi = \eta$, however it is not necessarily a distance. Just as for the Lorentzian distance on $C(\Lambda)$, we have a weak form of the triangle inequality which will be of some use.

Lemma 3.16. *There is a constant $\lambda \geq 1$ such that:*

$$\forall x \in C(\Lambda)\ \forall \xi, \eta, \tau \in \Lambda\ d_x(\xi,\eta) \leq \lambda(d_x(\xi,\tau) + d_x(\tau,\eta))$$

Proof. It is enough to show the inequality when ξ, η, τ are pairwise distinct ($\lambda = 1$ gives the inequality when it is not the case).

Consider the function $F : (x, \xi, \eta, \tau) \mapsto \frac{d_x(\xi,\tau) + d_x(\tau,\eta)}{d_x(\xi,\eta)}$ defined on $C(\Lambda) \times \Theta_3(\Lambda)$ where $\Theta_3(\Lambda)$ is the set of distinct triples of points of Λ. This function is Γ-invariant, and the action of Γ on $\Theta_3(\Lambda)$ is co-compact, so it is enough to see that $x \mapsto F(x, \xi, \eta, \tau)$ is bounded from below for fixed $(\xi, \eta, \tau) \in \Theta_3(\Lambda)$. Assume that it is not the case, then one can find a sequence $x_k \in C(\Lambda)$ such that $F(x_k, \xi, \eta, \tau) \to 0$. The expression of F is:

$$F(x, \xi, \eta, \tau) = \sqrt{\frac{\langle \xi|\tau\rangle\langle \eta|x\rangle}{\langle \xi|\eta\rangle\langle \tau|x\rangle}} + \sqrt{\frac{\langle \eta|\tau\rangle\langle \xi|x\rangle}{\langle \xi|\eta\rangle\langle \tau|x\rangle}}$$

The fact that $F(x_k, \xi, \eta, \tau) \to 0$ implies that $x_k \to \eta$ (first term of F) and $x_k \to \xi$ (second term of F). This is impossible because $\xi \neq \eta$.

$\square$

Given $x \in C(\Lambda)$ and $\xi, \eta \in \Lambda$, the quantity $d_x(\xi, \eta)$ can be computed from the lenghts of the side of any triangle whose vertices are x, a point of $[x\xi)$ and a point of $[x\eta)$. We want to stress out that the following lemma can be seen as a purely hyperbolic geometry result, since all the points are on a unique $\mathbb{H}^2 \subset \mathrm{AdS}^n$.

Lemma 3.17. *Let $x, y, z \in C(\Lambda)$ and $\xi, \eta \in \Lambda$. If $y \in [x\eta)$ and $z \in [x\xi)$, then:*

$$d_x(\xi, \eta)^2 = \frac{\cosh d(y,z) - \cosh(d(x,y) - d(x,z))}{2 \sinh d(x,y) \sinh d(x,z)}.$$

Proof. We denote by $\xi(u)$ (resp. $\eta(u)$) the geodesic joining x and ξ (resp. η). The equations are:

$$\xi(u) = e^{-u} x - \frac{\sinh u}{\langle x|\xi\rangle} \xi$$
$$\eta(u) = e^{-u} x - \frac{\sinh u}{\langle x|\eta\rangle} \eta$$

Since we have $y = \eta(u)$ (where $u = d(x,y)$) and $z = \xi(v)$ (where $v = d(x,z)$), we find:

$$\begin{aligned} -\cosh d(y,z) &= \langle y|z\rangle \\ &= \langle e^{-u}x - \frac{\sinh u}{\langle x|\eta\rangle}\eta | e^{-v}x - \frac{\sinh v}{\langle x|\xi\rangle}\xi\rangle \\ &= -e^{-u-v} - e^{-v}\sinh u - e^{-u}\sinh v + \sinh u \sinh v \frac{\langle \xi|\eta\rangle}{\langle \xi|x\rangle\langle x|\eta\rangle} \\ &= -\cosh(u-v) - 2 d_x(\xi,\eta)^2 \sinh u \sinh v \end{aligned}$$

$\square$

Corollary 3.18. *Let $\xi \in \Lambda$ and $r \in (0,1)$. If $y \in [x\xi)$ is such that $d(x,y) = -\ln r$, then $B_x(\xi, r) \subset \mathcal{S}_{\ln 6}(x,y)$.*

Proof. Let $\eta \in B_x(\xi, r)$, and let $z \in [x\eta)$ be such that $d(x,z) = d(x,y) = -\ln r$. We find $\cosh d(y,z) = 1 + 2(\sinh d(x,y) d_x(\xi,\eta))^2 \leq 3$, hence $d(y,z) \leq \mathrm{Argcosh}\, 3 \leq \ln 6$, and $\eta \in \mathcal{S}_{\ln 6}(x,y)$. $\square$

Corollary 3.19. *Let $\xi \in \Lambda$, $r \in (0,1)$ and $t > 0$. If $y \in [x\xi)$ is such that $d(x,y) = t + k + \frac{\ln 2}{2} - \ln \frac{r}{4}$, then $\mathcal{S}_t(x,y) \subset B_x(\xi, r)$.*

Proof. Let $y \in [x\xi)$ and $\eta \in \mathcal{S}_t(x,y)$. Given $z \in [x\eta)$ such that $d(z,y) < t$, we find:

$$d_x(\xi,\eta)^2 = \frac{\cosh d(y,z) - \cosh(d(x,y) - d(x,z))}{2\sinh d(x,y)\sinh d(x,z)} < \frac{e^t}{\sinh d(x,y)\sinh d(x,z)}$$

Since $d(x,z) \geq d(x,y) - t - k$, we find that $d(x,z) \geq \frac{\ln 2}{2}$, and $d(x,y) \geq \frac{\ln 2}{2}$, hence $\sinh d(x,y) \geq \frac{e^{d(x,y)}}{4}$ and $\sinh d(x,z) \geq \frac{e^{d(x,z)}}{4}$ (here we use the fact that $u \geq \frac{\ln 2}{2}$ implies $\sinh u \geq \frac{e^u}{4}$). Finally,

$$d_x(\xi,\eta)^2 < 16e^{t-d(x,y)-d(x,z)} \leq 16e^{2t+k-2d(x,y)}$$

In order to have $\mathcal{S}_t(x,y) \subset B_x(\xi,r)$, it is enough to have $4e^{t+\frac{k}{2}}e^{-d(x,y)} \leq r$, which is guaranteed by the condition on $d(x,y)$. ☐

Note that radial convergence can be expressed in terms of the Gromov distance d_x.

Lemma 3.20. *Let $\xi \in \Lambda$, and let γ_p be a sequence in Γ such that $\gamma_p.x$ converges radially to ξ. Denote by $\eta_p \in \Lambda$ a point that is causally related in $\partial\mathrm{AdS}^{n+1} \setminus \partial x^*$ to the endpoint of the half-geodesic based at x passing through $\gamma_p.x$. Then the sequence $e^{d(x,\gamma_p.x)}d_x(\xi,\eta_p)$ is bounded.*

Proof. Let $y_p \in [x\xi)$ be the point such that $d(x,y_p) = d(x,\gamma_p.x)$. Since $\gamma_p.x$ converges radially to ξ, we have that $d(\gamma_p.x, y_p)$ is bounded. Let z_p be the point on $[x\eta_p)$ such that $d(x,z_p) = d(x,\gamma_p.x)$.
We saw in the proof of Theorem 4.5 that $\gamma_p.x$ is causally related to z_p. By Lemma 3.17, we see that $e^{d(x,\gamma_p.x)}d_x(\xi,\eta_p) \leq K\sqrt{\cosh d(y_p,z_p)}$ for some constant K. Since z_p is causally related to $\gamma_p.x$, we also have that $d(y_p,z_p)$ is bounded, hence the result. ☐

3.6 Cross-ratios

In the last section, we will use the following lemma proven by J-P. Otal [Ota92] for Hadamard spaces. Recall that the cross ratio of four boundary points is defined by

$$[a,b,c,d] := \frac{d_x(a,c)d_x(b,d)}{d_x(a,d)d_x(b,c)}. \tag{1}$$

It is independent of x. Indeed

$$\begin{aligned}\left(\frac{d_x(a,c)d_x(b,d)}{d_x(a,d)d_x(b,c)}\right)^2 &= \frac{-\langle a\,|\,c\rangle}{\langle a\,|\,x\rangle\langle c\,|\,x\rangle}\frac{-\langle b\,|\,d\rangle}{\langle b\,|\,x\rangle\langle d\,|\,x\rangle}\frac{\langle a\,|\,x\rangle\langle d\,|\,x\rangle}{-\langle a\,|\,d\rangle}\frac{\langle b\,|\,x\rangle\langle c\,|\,x\rangle}{-\langle b\,|\,c\rangle} \\ &= \frac{\langle a\,|\,c\rangle\langle b\,|\,d\rangle}{\langle b\,|\,c\rangle\langle a\,|\,d\rangle}\end{aligned}$$

Lemma 3.21. *Let $\gamma \in \Gamma$ where $\Gamma \subset \mathrm{SO}(2,n)$ is a quasi-Fuchsian group. If $\gamma^-, \gamma^+ \in \Lambda$ are its repulsive and attractive fixed points, then for any $\xi \in \Lambda \setminus \{g^\pm\}$:*

$$[\gamma^-, \gamma^+, \gamma(\xi), \xi]_{AdS} = e^{\ell_{AdS}(\gamma)}.$$

Proof. By definition

$$[\gamma^-, \gamma^+, \gamma(\xi), \xi]^2_{AdS} = \frac{\langle \gamma^- \mid \gamma(\xi)\rangle \langle \gamma^+ \mid \xi \rangle}{\langle \gamma^+ \mid \gamma(\xi)\rangle \langle \gamma^- \mid \xi \rangle}$$

Let P be the plane in $\mathbb{R}^{2,n}$ such that $P \cap \mathrm{AdS} = (\gamma^-, \gamma^+)$. It is of signature $(+,-)$ and hence it's orthogonal $P^\perp$ satisfies $P \oplus P^\perp = \mathbb{R}^{2,n}$. It is clear from the definition of $\beta_{\gamma^\pm}$ that for $x \in (\gamma^-, \gamma^+)$, we have

$$\frac{\langle \gamma^- \mid \gamma(x)\rangle}{\langle \gamma^- \mid x\rangle} = e^{\beta_{\gamma^-}(\gamma(x),x)} = e^{\ell_{\mathrm{AdS}}(\gamma)}.$$

$$\frac{\langle \gamma^+ \mid x\rangle}{\langle \gamma^+ \mid \gamma(x)\rangle} = e^{\beta_{\gamma^+}(x,\gamma(x))} = e^{\ell_{\mathrm{AdS}}(\gamma)}.$$

Let $h \in P^\perp$ then

$$\frac{\langle \gamma^- \mid \gamma(x+h)\rangle}{\langle \gamma^- \mid x+h\rangle} = \frac{\langle \gamma^- \mid \gamma(x)\rangle}{\langle \gamma^- \mid x\rangle},$$

since γ preserve $P^\perp$. Hence for all $x \in \mathbb{R}^{2,n}$ we have

$$\beta_{\gamma^-}(\gamma(x), x) = \ell_{\mathrm{AdS}}(\gamma).$$

And by the same argument

$$\beta_{\gamma^+}(x, \gamma(x)) = \ell_{\mathrm{AdS}}(\gamma).$$

It follows by projectivizing that

$$\frac{\langle \gamma^- \mid \gamma(x)\rangle}{\langle \gamma^- \mid x\rangle} = \frac{\langle \gamma^- \mid \gamma(\xi)\rangle}{\langle \gamma^- \mid \xi\rangle} = e^{\ell_{AdS}(\gamma)}.$$

Therefore

$$[\gamma^-, \gamma^+, \gamma(\xi), \xi]^2_{AdS} = e^{2\ell_{\mathrm{AdS}}(\gamma)}$$

□

The following lemma is an adaptation of a result in Bourdon [Bou96].

Lemma 3.22. *Let $\xi_1, \xi_2, \eta_1, \eta_2 \in \Lambda$. The geodesics $(\xi_1\xi_2)$ and $(\eta_1\eta_2)$ intersect if and only if $[\xi_1, \eta_1, \eta_2, \xi_2] + [\xi_1, \eta_2, \eta_1, \xi_2] = 1$.*

Proof. If the geodesics intersect, the four points $\xi_1, \xi_2, \eta_1, \eta_2$ lay on the boundary of a totally geodesic copy of $\mathbb{H}^2$, and it follows from the properties of the usual cross-ratio on the circle that $[\xi_1, \eta_1, \eta_2, \xi_2] + [\xi_1, \eta_2, \eta_1, \xi_2] = 1$.
Now assume that $[\xi_1, \eta_1, \eta_2, \xi_2] + [\xi_1, \eta_2, \eta_1, \xi_2] = 1$. First, we will show that there is a point $x \in (\xi_1 \xi_2)$ such that $d_x(\xi_1, \eta_1) = d_x(\xi_2, \eta_2)$.
A point $x \in (\xi_1 \xi_2)$ can be written $x = a\xi_1 + -\frac{1}{2a\langle \xi_1 | \xi_2 \rangle}\xi_2$ for some positive real number a.
Let us compute $d_x(\xi_1, \eta_1)$ and $d_x(\xi_2, \eta_2)$.

$$\begin{aligned}
\langle x | \xi_1 \rangle &= \frac{-1}{2a} \\
\langle x | \xi_2 \rangle &= a\langle \xi_1 | \xi_2 \rangle \\
\langle x | \eta_1 \rangle &= a\langle \xi_1 | \eta_1 \rangle - \frac{\langle \eta_1 | \xi_2 \rangle}{2a\langle \xi_1 | \xi_2 \rangle} \\
\langle x | \eta_2 \rangle &= a\langle \xi_1 | \eta_2 \rangle - \frac{\langle \eta_2 | \xi_2 \rangle}{2a\langle \xi_1 | \xi_2 \rangle}
\end{aligned}$$

$$\begin{aligned}
\left(\frac{d_x(\xi_1, \eta_1)}{d_x(\xi_2, \eta_2)} \right)^2 &= \frac{\langle \xi_1 | \eta_1 \rangle \langle x | \xi_2 \rangle \langle x | \eta_2 \rangle}{\langle x | \xi_1 \rangle \langle x | \eta_1 \rangle \langle \xi_2 | \eta_2 \rangle} \\
&= \frac{\langle \xi_1 | \eta_1 \rangle a \langle \xi_1 | \xi_2 \rangle \left(a\langle \xi_1 | \eta_2 \rangle - \frac{\langle \eta_2 | \xi_2 \rangle}{2a\langle \xi_1 | \xi_2 \rangle} \right)}{\frac{-1}{2a} \left(a\langle \xi_1 | \eta_1 \rangle - \frac{\langle \eta_1 | \xi_2 \rangle}{2a\langle \xi_1 | \xi_2 \rangle} \right) \langle \xi_2 | \eta_2 \rangle} \\
&= \frac{-\langle \xi_1 | \eta_1 \rangle \langle \xi_1 | \xi_2 \rangle}{2\langle \xi_2 | \eta_2 \rangle} \frac{a^4 \langle \xi_1 | \eta_2 \rangle - a^2 \frac{\langle \eta_2 | \xi_2 \rangle}{2\langle \xi_1 | \xi_2 \rangle}}{a^2 \langle \xi_1 | \eta_1 \rangle - \frac{\langle \eta_1 | \xi_2 \rangle}{2\langle \xi_1 | \xi_2 \rangle}}
\end{aligned}$$

We see from this formula that $\frac{d_x(\xi_1, \eta_1)}{d_x(\xi_2, \eta_2)}$ goes to 0 as a goes to 0, and to $+\infty$ as a goes to $+\infty$, so by continuity it follows that there is $a > 0$ such that this value is 1, i.e. there is $x \in (\xi_1 \xi_2)$ such that $d_x(\xi_1, \eta_1) = d_x(\xi_2, \eta_2)$.

Notice that ξ_1, ξ_2, η_1 lie in the boundary of a totally geodesic copy of $\mathbb{H}^2$. It follows from the description of d_x on the boundary of $\mathbb{H}^2$ and Pythagora's Theorem that $d_x(\xi_1, \eta_1)^2 + d_x(\xi_2, \eta_1)^2 = 1$. A similar formula is valid for η_2, which gives us $d_x(\xi_1, \eta_2) = d_x(\xi_2, \eta_1)$.

The fact that $[\xi_1, \eta_1, \eta_2, \xi_2] + [\xi_1, \eta_2, \eta_1, \xi_2] = 1$ translates as:

$$\frac{d_x(\xi_1, \eta_2) d_x(\eta_1, \xi_2)}{d_x(\xi_1, \xi_2) d_x(\eta_1, \eta_2)} + \frac{d_x(\xi_1, \eta_1) d_x(\eta_2, \xi_2)}{d_x(\xi_1, \xi_2) d_x(\eta_1, \eta_2)} = 1$$

We can rewrite this expression to compute $d_x(\eta_1, \eta_2)$:

$$\begin{aligned}
d_x(\eta_1, \eta_2) &= d_x(\xi_1, \eta_1) d_x(\eta_2, \xi_2) + d_x(\xi_1, \eta_2) d_x(\eta_1, \xi_2) \\
&= d_x(\xi, \eta_1)^2 + d_x(\eta_1, \xi_2)^2 \\
&= 1
\end{aligned}$$

This implies that $x \in (\xi_1\xi_2) \cap (\eta_1\eta_2)$, hence the result.

□

4 Conformal densities

In this section we define conformal densities and prove that the Patterson-Sullivan construction with the Lorentzian distance d on $C(\Lambda)$ gives one of dimension $\delta(\Gamma)$, Proposition 4.3. Then we study the properties of conformal densities and establish the so-called Shadow Lemma 4.5. We ends by proving the uniqueness of conformal density, Theorem 4.16.

The proofs are generally similar to the hyperbolic case, up to taking care of the additive constant k from the triangle inequality. However technical difficulties sometimes appear due to the Lorentzian context, this is notably the case for the Shadow lemma, and Vitali's lemma for shadows.

We will mainly follow the notes of J.-F. Quint [Qui06]. Another reference for this notion in Hyperbolic geometry is the book of P. Nicholls [Nic89].

Definition 4.1. *A conformal density of dimension s is a family of measures $(\nu_x)_{x\in C(\Lambda)}$ satisfying the following conditions:*

1. $\forall g \in \Gamma$, $g^*\nu_x = \nu_{gx}$ *(where $g^*\nu(E) = \nu(g^{-1}E)$)*
2. $\forall x, y \in D$, $\frac{d\nu_x}{d\nu_y}(\xi) = e^{-s\beta_\xi(x,y)}$
3. $\mathrm{supp}\,(\nu_x) = \Lambda$

4.1 Existence of conformal densities

Let P be the Poincaré series associated to Γ :

$$P(s) = \sum_{\gamma\in\Gamma} e^{-sd(\gamma o, o)}.$$

As in the hyperbolic case, we don't know in advance that P diverges at $\delta(\Gamma)$. In order to solve this problem, S. J.Patterson proposed a modification of this Poincaré series using an auxiliary function h

Lemma 4.2. *[Pat76] Let $\sum_{n\in\mathbb{N}} a_n^{-s}$ be a Dirichlet series, of convergence radius equal to δ. There exists an increasing function $h : [0, \infty[\to [0, \infty[$ such that :*

- *The radius of convergence of the series $\sum k(a_n)a_n^{-s}$ is δ. The series diverges at δ.*
- *For all $\varepsilon > 0$ there exists $y_0 > 0$ such that for all $y > y_0$, $x > 1$ we have :*

$$h(y + x) < x^{\varepsilon} h(y).$$

We called Q the modified Poincaré series :

$$Q(s) = \sum_{\gamma \in \Gamma} h(d(\gamma o, o)) e^{-sd(\gamma o, o)}.$$

We denote Δ_x the Dirac mass at x. We consider the following family of measures for every $x \in C(\Lambda)$, $s > \delta$:

$$\mu_x^s = \frac{\sum_{\gamma \in \Gamma} h(d(\gamma o, x)) e^{-sd(\gamma o, x)} \Delta_{\gamma o}}{Q(s)}.$$

They admits converging subsequence as $s \to \delta$ for weak topology, denoted by μ_x. The usual remark also stands : there could a priori exist different weak limit but we will see that it is in fact unique and will be called the Patterson-Sullivan density.

Proposition 4.3. *μ_x is a conformal density of dimension δ.*

Proof. Let us show the invariance

$$\begin{aligned} g^* \mu_x^s(E) &= \mu_x^s(g^{-1}E) \\ &= \frac{\sum_{\gamma \in \Gamma} h(d(\gamma o, x)) e^{-sd(\gamma o, x)} \Delta_{\gamma o}(g^{-1}E)}{Q(s)} \\ &= \frac{\sum_{\gamma \in \Gamma} h(d(\gamma o, x)) e^{-sd(\gamma o, x)} \Delta_{g\gamma o}(E)}{Q(s)} \\ &= \frac{\sum_{\gamma' \in \Gamma} h(d(\gamma' o, gx)) e^{-sd(\gamma' o, gx)} \Delta_{\gamma' o}(E)}{Q(s)} \\ &= \mu_{gx}^s(E). \end{aligned}$$

By passing to the limit we obtain the invariance of μ_x by Γ.

Let $\epsilon > 0$. Let $N(\xi) \subset \overline{\mathrm{AdS}}$ be a neighborhood of $\xi \in \Lambda$, such that $|\beta_\xi(x, y) - (d(z, x) - d(z, y))| \leq \epsilon$ for $z \in N(\xi)$. We have

$$\begin{aligned} \mu_x^s(N_\xi) &= \frac{1}{Q(s)} \sum_{\gamma o \in N(\xi)} h(d(\gamma o, x)) e^{-sd(\gamma o, x)} \\ &\leq \frac{1}{Q(s)} e^{s\epsilon} e^{-s\beta_\xi(x,y)} \sum_{\gamma o \in N(\xi)} h(d(\gamma o, x)) e^{-sd(\gamma o, y)} \end{aligned}$$

Patterson function h is increasing and $d(\gamma o, x) \leq d(\gamma o, y) + K$ hence

$$\mu_x^s(N_\xi) \leq \frac{1}{Q(s)} e^{s\epsilon} e^{-s\beta_\xi(x,y)} \sum_{\gamma o \in N(\xi)} h(d(\gamma o, y) + K) e^{-sd(\gamma o, y)}$$

By the second property of the Patterson function and the fact that $d(\gamma o, x) \to \infty$ as $\gamma o \to \xi$ we have :

$$\begin{aligned} \mu_x^s(N_\xi) &\leq \frac{1}{Q(s)} e^{s\epsilon} e^{-s\beta_\xi(x,y)} e^{\epsilon K} \sum_{\gamma o \in N(\xi)} h(d(\gamma o, y)) e^{-sd(\gamma o, y)} \\ &\leq e^{s\epsilon} e^{\epsilon K} e^{-s\beta_\xi(x,y)} \mu_y^s(N_\xi) \end{aligned}$$

By letting $\epsilon \to 0$ and $s \to \delta$ we get

$$\frac{d\mu_x}{d\mu_y}(\xi) \leq e^{-s\beta_\xi(x,y)}$$

The same computation, switching the role of x and y gives the other inequality. Hence we obtain the quasi-conformal relation

$$\frac{d\mu_x}{d\mu_y}(\xi) = e^{-s\beta_\xi(x,y)} \tag{2}$$

Since $Q(s) \to \infty$ as $s \to \delta$, the support of μ_x is included in Λ. Moreover from the quasi-conformal relation μ_x and μ_y have the same support. Recall that the action of Γ on Λ is minimal, therefore thanks to the invariance by Γ, this implies that the support is exactly Λ.

□

4.2 Properties of conformal densities

4.2.1 Atomic part

Proposition 4.4. *Let ν_x be a conformal density, then for all $x \in C(\Lambda)$, ν_x has no atom.*

Proof. Assume by contradiction that $\exists \xi$, $\nu_x(\xi) > 0$. First, let us assume there exists $g \in \Gamma$ such that $g\xi = \xi$. We have

$$\begin{aligned} \nu_x(\xi) &= \nu_x(g^i\xi) \\ &= \nu_{g^{-i}x}(\xi) \\ &= e^{-\delta\beta_\xi(g^{-i}x,x)} \nu_x(\xi) \end{aligned}$$

As ξ is assume to be fixed by g we have $\beta_\xi(g^{-i}x, x) = \beta_\xi(x, g^i x)$, and then $\lim_{n\to\infty} e^{-\delta\beta_\xi(g^{-i}x,x)} = +\infty$. Therefore we can assume that $Stab_\Gamma(\xi) = \mathrm{Id}$,

$$\begin{aligned} \nu_x(\Lambda) &\geq \sum_{g\in\Gamma} \nu_x(g\xi) \\ &\geq \sum_{g\in\Gamma} e^{-\delta\beta_\xi(g^{-1}x,x)} \nu_x(\xi) \end{aligned}$$

Every limit point is radial, hence we can find a sequence $g_i \in \Gamma$ such that $\beta_\xi(g_i^{-1}x, x) \to \infty$ and therefore $\nu_x(\Lambda) = +\infty$. Contradiction. □

4.2.2 Shadow Lemma

Theorem 4.5 (Shadow lemma)**.** *Let ν be a conformal density of dimension s, and $x \in C(\Lambda)$. There is $r_0 > 0$ such that, for all $r > r_0$, there is $C(r) > 0$ satisfying:*

$$\frac{1}{C(r)} e^{-sd(x,\gamma.x)} \leq \nu_x(\mathcal{S}_r(x, \gamma.x)) \leq C(r) e^{-sd(x,\gamma.x)}$$

For this we follow the proof of J.-F. Quint [Qui06].

Lemma 4.6. *[Qui06, Lemma 4.2] For any $x \in C(\Lambda)$, we have:*

$$\sup\{d_x(\xi, \eta) | \xi, \eta \in \Lambda, \exists y \in C(\Lambda) \cap J([x\eta]), \xi \notin \mathcal{S}_r(y, x)\} \xrightarrow[r \to +\infty]{} 0$$

Proof. Assume that the result is false, so we can find $\varepsilon > 0$, $r_k \to +\infty$, $\eta_k \in \Lambda$, $y_k \in C(\Lambda)$ with $y_k \in J([x\eta_k])$ and $\xi_k \notin \mathcal{S}_{r_k}(y_k, x)$ such that $d_x(\xi_k, \eta_k) \geq \varepsilon$ for all $k \in \mathbb{N}$.
Since $d(x, y_k) \geq r_k$, the sequence y_k leaves every compact set of $C(\Lambda)$, and up to a subsequence, we may assume that there are $\xi, \eta \in \Lambda$ such that $\xi_k \to \xi$ and $y_k \to \eta$.
Note that one also has $\eta_k \to \eta$ (since a limit point of η_k has to be causally related to η, and the limit set is acausal), hence $d_x(\xi, \eta) \geq \varepsilon$.

Let z be a point on the geodesic $(\eta\xi)$ joining ξ with η. Since the geodesics $(y_k\xi_k)$ joining y_k with ξ_k converge to $(\eta\xi)$, for k sufficiently large, $(y_k\xi_k)$ interesects any neighbourhood of z, in particular $B(x, d(x, z) + 1)$. However for k large enough, one also has $r_k > d(x, z) + 1$, wich implies that $\xi_k \in \mathcal{S}_{r_k}(y_k, x)$. This is a contradiction.

□

Lemma 4.7. *Let ν be a conformal density of dimension s, and $x \in C(\Lambda)$. There is $\varepsilon > 0$ such that:*

$$\forall \xi \in \Lambda,\ \nu_x(\Lambda \setminus B_x(\xi, \varepsilon)) \geq \varepsilon.$$

Proof. By Proposition 4.4, the measure ν_x has no atoms. Let $\xi \in \Lambda$. Since $0 = \nu_x(\{\xi\}) = \lim_{r \to 0} \nu_x(B_x(\xi, r))$, we can find $r_\xi > 0$ such that $\nu_x(B_x(\xi, r_\xi)) < \nu_x(\Lambda)$.

Let $\lambda \geq 1$ be given by Lemma 3.16 (i.e. a multiplicative constant in the triangle inequality for d_x). Since Λ is compact, we can find $\xi_1, \ldots, \xi_k \in \Lambda$ such that $\Lambda = \bigcup_{i=1}^k B_x(\xi, \frac{r_i}{2\lambda})$, where $r_i = r_{\xi_i}$. Let $r = \min \frac{r_i}{2\lambda}$. If $\xi \in \Lambda$, there is $i \in \{1, \ldots, k\}$ with $\xi \in B(\xi_i, \frac{r_i}{2\lambda})$, which implies $B_x(\xi, r) \subset B_x(\xi_i, r_i)$.

By letting $\alpha = \min_i \nu_x(\Lambda) - \nu_x(B_x(\xi_i, r_i))$, we see that $\varepsilon = \min(r, \alpha)$ fulfils the requirements.

□

Proof of Theorem 4.5. Let $r > 0$ and $\gamma \in \Gamma$.

$$\begin{aligned}\nu_x(\mathcal{S}_r(x, \gamma.x)) &= \nu_x(\gamma \mathcal{S}_r(\gamma^{-1}.x, x)) \\ &= \nu_{\gamma^{-1}.x}(\mathcal{S}_r(\gamma^{-1}.x, x)) \\ &= \int_{\mathcal{S}_r(\gamma^{-1}.x,x)} e^{-sb_\xi(\gamma^{-1}.x,x)} d\nu_x(\xi)\end{aligned}$$

If $s \geq 0$, Lemma 3.10 now implies that:

$$\nu_x(\mathcal{S}_r(\gamma^{-1}.x, x))e^{-sk}e^{-sd(x,\gamma.x)} \leq \nu_x(\mathcal{S}_r(x, \gamma.x)) \leq \nu_x(\Lambda)e^{2s(r+k)}e^{-sd(x,\gamma.x)}$$

If $s < 0$, we get:

$$\nu_x(\mathcal{S}_r(\gamma^{-1}.x, x))e^{-2s(r+k)}e^{-sd(x,\gamma.x)} \leq \nu_x(\mathcal{S}_r(x, \gamma.x)) \leq \nu_x(\Lambda)e^{sk}e^{-sd(x,\gamma.x)}$$

All we have left to show is that there are $\varepsilon > 0$ and $r_0 > 0$ such that $\nu_x(\mathcal{S}_r(\gamma^{-1}.x, x)) \geq \varepsilon$ for all $r > r_0$ and $\gamma \in \Gamma$.
Let $\varepsilon > 0$ be given by Lemma 4.7. By Lemma 4.6, there is $r_0 \geq 0$ such that, for all $r \geq r_0$, one has:

$$\sup\{d_x(\xi, \eta) | \xi, \eta \in \Lambda, \exists y \in C(\Lambda) \cap J([x\eta]), \xi \notin \mathcal{S}_r(y, x)\} \leq \varepsilon \qquad (3)$$

Let $\gamma \in \Gamma$, and denote by $\tau \in \partial\mathrm{AdS}^{n+1}$ the endpoint of the geodesic going from x to $\gamma^{-1}.x$. Denote by η a point of Λ wich is in the future or past of τ in $\partial\mathrm{AdS}^{n+1} \setminus \partial x^*$ (such a point exists because Λ is a Cauchy hypersurface in $\partial\mathrm{AdS}^{n+1} \setminus \partial x^*$. By letting $z \in [x\eta)$ be the point such that $d(x, z) = d(x, \gamma^{-1}.x)$, we can compute $\langle z|\gamma^{-1}.x\rangle$. If $t = d(x, \gamma^{-1}.x) = d(x, z)$, we have:

$$\begin{aligned}\gamma^{-1}.x &= e^{-t}x - \frac{\sinh t}{\langle x|\tau\rangle}\tau \\ z &= e^{-t}x - \frac{\sinh t}{\langle x|\eta\rangle}\eta\end{aligned}$$

This leads to:

$$\begin{aligned}\langle z|\gamma^{-1}.x\rangle &= \langle e^{-t}x - \frac{\sinh t}{\langle x|\tau\rangle}\tau | e^{-t}x - \frac{\sinh t}{\langle x|\eta\rangle}\eta\rangle \\ &= e^{-2t}\langle x|x\rangle - e^{-t}\sinh t\frac{\langle x|\tau\rangle}{\langle x|\tau\rangle} - e^{-t}\sinh t\frac{\langle x|\eta\rangle}{\langle x|\eta\rangle} + (\sinh t)^2\frac{\langle \tau|\eta\rangle}{\langle x|\tau\rangle\langle x|\eta\rangle} \\ &= -1 + (\sinh t)^2\frac{\langle \tau|\eta\rangle}{\langle x|\tau\rangle\langle x|\eta\rangle}\end{aligned}$$

Since η and τ are causally related, we have $(\sinh t)^2\frac{\langle \tau|\eta\rangle}{\langle x|\tau\rangle\langle x|\eta\rangle} \geq 0$, hence $\langle z|\gamma^{-1}.x\rangle \geq -1$, i.e. z and $\gamma^{-1}.x$ are causally related, and $\gamma^{-1}.x \in J([x\eta))$. The inequality 3 shows that $\mathcal{S}_r(\gamma^{-1}.x, x) \supset \Lambda \setminus B_x(\eta, \varepsilon)$. Now recall that ε was chosen such that $\nu_x(\Lambda \setminus B_x(\eta, \varepsilon)) \geq \varepsilon$, wich concludes the proof.

□

4.2.3 Shadow and balls

Using the convex cocompactness, we can generalise Theorem 4.5 to all shadows.

Theorem 4.8. *Let ν be a conformal density of dimension s, and let $x \in C(\Lambda)$. There is $r_0' > 0$ such that for all $r \geq r_0'$, there is a constant $C'(r) > 0$ satisfying:*

$$\frac{1}{C'(r)} e^{-sd(x,y)} \leq \nu_x(\mathcal{S}_r(x,y)) \leq C'(r) e^{-sd(x,y)}$$

for all $y \in C(\Lambda)$.

Proof. Let $r_0 > 0$ be given by Theorem 4.5. The action of Γ on $C(\Lambda)$ being compact, we can choose $R > 0$ such that the ball $B(x,R)$ contains a fundamental domain for the action on $C(\Lambda)$. We will show that $r_0' = r_0 + R + k$ and $C'(r) = \max(C(r+R+k), C(r-R-k)) e^{s(R+k)}$ fulfil the requirements.
Let $y \in C(\Lambda)$ and $r > r_0'$. There is $\gamma \in \Gamma$ such that $\gamma^{-1}.y \in B(x,R)$, hence $d(y,\gamma.x) < R$. Let $\xi \in \mathcal{S}_r(x,y)$, and $z \in [x\xi)$ such that $d(y,z) < r$. We find $d(\gamma.x, z) \leq d(\gamma.x, y) + d(y,z) + k < R + r + k$. It follows that $\mathcal{S}_r(x,y) \subset \mathcal{S}_{r+R+k}(x,\gamma.x)$.
A similar computation shows that $\mathcal{S}_r(x,y) \supset \mathcal{S}_{r-R-k}(x,\gamma.x)$.
By Theorem 4.5, we get:

$$\begin{array}{ccc} \nu_x(\mathcal{S}_{r-R-k}(x,\gamma.x)) \leq & \nu_x(\mathcal{S}_r(x,y)) \leq & \nu_x(\mathcal{S}_{r+R+k}(x,\gamma.x)) \\ \frac{1}{C(r-R-k)} e^{-\delta_\Gamma d(x,\gamma.x)} \leq & \nu_x(\mathcal{S}_r(x,y)) \leq & C(r+R+k) e^{-\delta_\Gamma d(x,\gamma.x)} \end{array}$$

We also have $d(x,\gamma.x) \leq d(x,y) + d(y,\gamma.x) + k \leq d(x,y) + R + k$, as well as $d(x,\gamma.x) \geq d(x,y) - R - k$. It follows that:

$$\frac{e^{-s(R+k)}}{C(r-R-k)} e^{-sd(x,y)} \leq \nu_x(\mathcal{S}_r(x,y)) \leq e^{s(R+k)} C(r+R+k) e^{-sd(x,y)}$$

□

We follow Sullivan's work [Sul79] and show that the Shadow Lemma implies the following:

Theorem 4.9. *Let ν be a conformal density of dimension s, and let $x \in C(\Lambda)$. There is $c > 0$ such that for all $\xi \in \Lambda$, $r \in (0,1)$, we have:*

$$\frac{\nu_x(B_x(\xi,r))}{r^s} \in \left[\frac{1}{c}, c\right]$$

Proof. Let $r_0' > 0$ be given by Theorem 4.8, and $t = \max(r_0', \ln 6)$.
Let $\xi \in \Lambda$ and $r \in (0,1)$. According to Corollary 3.18, by letting $y_1 \in [x\xi)$

be the point such that $d(x, y_1) = -\ln r$, we find $B_x(\xi, r) \subset \mathcal{S}_{\ln 6}(x, y_1) \subset \mathcal{S}_t(x, y_1)$. By Theorem 4.8, we get $\nu_x(B_x(\xi, r)) \leq C'(t)e^{-sd(x,y_1)} = C'(t)r^s$. Now let $y_2 \in [x\xi)$ be such that $d(x, y_2) = t + k + \frac{\ln 2}{2} - \ln \frac{r}{4}$. According to Corollary 3.19, we have the inclusion $\mathcal{S}_t(x, y_2) \subset B_x(\xi, r)$. Theorem 4.8 now gives us $\nu_x(B_x(\xi, r)) \geq \frac{1}{C'(t)}e^{-sd(x,y_2)} \geq \frac{1}{c}r^s$ where $c = C'(t)e^{s(t+k+\frac{5\ln 2}{2})}$. □

4.3 Uniqueness of conformal densities

4.3.1 Uniqueness of dimension

We follow the notes of Quint [Qui06] and adapt it to our setting.

Proposition 4.10. *If there is a non trivial conformal density of dimension s, then $s \geq \delta(\Gamma)$.*

Proof. Let ν be a non trivial conformal density of dimension s. First, note that the left hand side of the inequality in the Shadow Lemma implies that $s \geq 0$: if it were not the case, the measure ν_x would be infinite.
Let C and r_0 be given by Theorem 4.5, and let $r \geq r_0$. For $i \in \mathbb{N}$, we set $\Gamma_i = \{\gamma \in \Gamma | i \leq d(x, \gamma.x) < i + 1\}$, and a_i the number of elements of Γ_i.
Let p be the cardinal of $Z = \{\gamma \in \Gamma | d(x, \gamma.x) \leq 1 + 4(r + k)\}$. We will show that given $\xi \in \Lambda$ and $i \in \mathbb{N}$, there are at most p elements $\gamma \in \Gamma_i$ such that $\xi \in \mathcal{S}_r(x, \gamma.x)$.
Let $\gamma, \gamma' \in \Gamma_i$ be such that $\xi \in \mathcal{S}_r(x, \gamma.x) \cap \mathcal{S}_r(x, \gamma'.x)$. There is $y \in [x\xi)$ such that $d(\gamma.x, y) \leq r$. Let us find an estimation for $d(x, y)$:

$$d(x, y) \leq d(x, \gamma.x) + d(\gamma.x, y) + k \leq i + 1 + r + k$$
$$d(x, y) \geq d(x, \gamma.x) - d(y, \gamma.x) - k \geq i - r - k$$

$$i - r - k \leq d(x, y) \leq i + 1 + r + k$$

Similarly there is $z \in [x\xi)$ such that $d(\gamma'.x, z) \leq r$ and:

$$i - r - k \leq d(x, z) \leq i + 1 + r + k$$

Since y and z lie in the same half geodesic $[x\xi)$, we see that

$$d(y, z) = \pm(d(x, y) - d(x, z))$$

This shows that $d(y, z) \leq 1 + 2(r + k)$. Finally, we find:

$$d(\gamma.x, \gamma'.x) \leq d(\gamma.x, y) + d(y, z) + d(z, \gamma'.x) + 2k \leq 1 + 4(r + k)$$

This means that $\gamma^{-1}\gamma' \in Z$, which shows the desired bound on the number of such elements of Γ_i.

$$
\begin{aligned}
\nu_x(\Lambda) &\geq \nu_x\left(\bigcup_{\gamma\in\Gamma_i} \mathcal{S}_r(x,\gamma.x)\right)\\
&\geq \frac{1}{p}\sum_{\gamma\in\Gamma_i} \nu_x\left(S_r(x,\gamma.x)\right)\\
&\geq \frac{1}{pC}\sum_{\gamma\in\Gamma_i} e^{-sd(x,\gamma.x)}\\
&\geq \frac{1}{pC}e^{-s(i+1)}a_i
\end{aligned}
$$

Let $D = pC\nu_x(\Lambda)$, so that we find $a_i \leq De^{s(i+1)}$ for all i, and :

$$
\frac{1}{i}\ln(a_0+\cdots+a_i) \leq \frac{1}{i}\ln(D(i+1)e^{s(i+1)}) \xrightarrow[i\to+\infty]{} s
$$

Since $\delta_\Gamma = \limsup \frac{1}{i}\ln(a_0+\cdots+a_i)$, we find that $s \geq \delta_\Gamma$.

□

Knowing the fact that every point of Λ is conical, we can turn the inequality of Corollary 4.10 into an equality.

Proposition 4.11. *If there is a conformal density of dimension s, then the Poincaré series diverges at s.*

Proof. We will use an enumeration $\Gamma = \{\gamma_p | p \in \mathbb{N}\}$. By Lemma 3.15, we have that:

$$
\Lambda \subset \bigcup_{r>0}\bigcap_{N\in\mathbb{N}}\bigcup_{p\geq N} \mathcal{S}_r(x,\gamma_p.x) \tag{4}
$$

Assume that $\sum_{p=0}^{+\infty} e^{-sd(x,\gamma_p.x)} < +\infty$, and let ν be a conformal density of dimension s.
Let r_0 be given by Theorem 4.5, and let $r \geq r_0$ and C the associated constant from the same theorem. Given $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that $\sum_{p=N}^{+\infty} e^{-sd(x,\gamma_p.x)} \leq \varepsilon$. By Theorem 4.5, we have that $\nu_x(\mathcal{S}_r(x,\gamma_p.x)) \leq Ce^{-sd(x,\gamma_p.x)}$ for all $p \in \mathbb{N}$, hence $\nu_x(\bigcup_{p\geq N}\mathcal{S}_r(x,\gamma_p.x)) \leq C\varepsilon$. This implies that $\nu_x(\bigcap_{N\in\mathbb{N}}\bigcup_{p\geq N}\mathcal{S}_r(x,\gamma_p.x)) = 0$.
If $r \leq r_0$, then $\mathcal{S}_r(x,\gamma_p.x) \subset \mathcal{S}_{r_0}(x,\gamma_p.x)$ for all $p \in \mathbb{N}$, so we also find $\nu_x(\bigcap_{N\in\mathbb{N}}\bigcup_{p\geq N}\mathcal{S}_r(x,\gamma_p.x)) = 0$.
Since the union over all $r > 0$ in 4 is increasing, it can be written as a countable union, and we find that $\nu_x(\Lambda) = 0$, which is a contradiction. Therefore $\sum_{p=0}^{+\infty} e^{-sd(x,\gamma_p.x)} = +\infty$. □

Corollary 4.12. *If there is a conformal density of dimension s, then $s = \delta_\Gamma$.*

Proof. Corollary 4.10 gives us $s \geq \delta_\Gamma$, and Proposition 4.11 implies that $s \leq \delta_\Gamma$. □

Corollary 4.13.
$$\sum_{\gamma\in\Gamma} e^{-\delta_\Gamma d(x,\gamma.x)} = +\infty$$

Proof. This is a straightforward consequence of Proposition 4.11 and the existence of the Patterson-Sullivan density which is of dimension δ_Γ. □

4.3.2 Ergodicity

In order to show ergodicity we need to show the existence of density points for shadows. Since it is classical in the hyperbolic setting and quite technical we postpone the proof to the appendix. The proof in the hyperbolic case can be found in [Nic89].

Lemma 4.14. $\forall x \in C(\Lambda)$, $\forall \epsilon > 0$, $\exists R > 0$, $\forall y \in C(\Lambda)$,

$$\nu_x(\mathcal{S}_R(x,y)) \geq \nu_x(\Lambda) - \epsilon.$$

Proof. Let fix $x \in C(\Lambda)$ and $\epsilon > 0$. From Lemma 1.5 of Shadow's Lemma, there exists R sufficiently large such that for all $\xi, \eta \in \mathcal{S}_R(x,y)$ we have $d_x(\xi,\eta) \leq \epsilon$. Hence $\Lambda \setminus \mathcal{S}_R(x,y) \subset B_x(\xi,\epsilon)$. This implies by Lemma 1.6 that

$$\begin{aligned} \nu_x(\Lambda \setminus \mathcal{S}_R(x,y)) &\leq \nu_x(B_x(\xi,\epsilon)) \\ \nu_x(\Lambda) - \nu_x(\mathcal{S}_R(x,y)) &\leq \epsilon \\ \nu_x(\mathcal{S}_R(x,y)) &\geq \nu_x(\Lambda) - \epsilon. \end{aligned}$$

□

Theorem 4.15. *[Nic89, Corollary 5.2.4] A conformal density is ergodic.*

Proof. Let ν_x be a conformal density. Let A be a Γ-invariant subset of Λ. Suppose that $\nu(A) > 0$ and let $\xi \in A$ be a density point and $\gamma_n x$ be a radial sequence converging to ξ, such that

$$\lim_{n\to\infty} \frac{\nu_x(\mathcal{S}_R(x,\gamma_n x) \cap A)}{\nu_x(\mathcal{S}_R(x,g_n x))} = 1.$$

Remark that for any borelian $E \subset \Lambda$ and element $\gamma \in \Gamma$ we have

$$\begin{aligned} \nu_x(\gamma^{-1}E) &= \nu_{\gamma x}(E) \\ &= \int_E e^{-\beta_\xi(\gamma x, x)} d\nu_x(\xi) \end{aligned}$$

Applying this to $\mathcal{S}_R(\gamma_n^{-1}x, x) \cap A = \gamma_n^{-1}\Big(\mathcal{S}_R(x,\gamma_n x) \cap A\Big)$ we have

$$\nu_x(\mathcal{S}_R(\gamma_n^{-1}x,x)\cap A) \;=\; \int_{\mathcal{S}_R(x,\gamma_n x)\cap A} e^{-\beta_\xi(\gamma_n x,x)} d\nu_x(\xi).$$

For all $\xi \in \mathcal{S}_R(x,\gamma_n x)$ it follows from Lemma 1.4 Shadow Lemma, that there exists $C > 0$

$$d(x,\gamma_n x) - C \leq -\beta_\xi(\gamma_n x,x) = \beta_\xi(x,\gamma_n x) \leq d(x,\gamma_n x) + C.$$

Hence

$$\begin{aligned}
\frac{\nu_x(\mathcal{S}_R(\gamma_n^{-1}x,x)\cap A)}{\nu_x(\mathcal{S}_R(\gamma_n^{-1}x,x))} &= 1 - \frac{\int_{\mathcal{S}_R(x,\gamma_n x)\cap A^c} e^{-\beta_\xi(\gamma_n x)} d\nu_x(\xi)}{\int_{\mathcal{S}_R(\gamma_n^{-1}x,x)} e^{-\beta_\xi(\gamma_n x)} d\mu_x(\xi)} \\
&\geq 1 - \frac{e^{C} e^{d(\gamma_n x,x)}}{e^{-C} e^{d(\gamma_n x,x)}} \frac{\int_{\mathcal{S}_R(x,\gamma_n x)\cap A^c} d\nu_x(\xi)}{\int_{\mathcal{S}_R(x,\gamma_n x)} d\mu_o(\xi)} \\
&\geq 1 - K \frac{\int_{\mathcal{S}_R(x,\gamma_n x)\cap A^c} d\nu_x(\xi)}{\int_{\mathcal{S}_R(x,\gamma_n x)} d\nu_x(\xi)} \\
&\geq 1 - K \frac{\nu_x(\mathcal{S}_R(x,\gamma_n x)\cap A^c)}{\nu_x(\mathcal{S}_R(x,\gamma_n x))}.
\end{aligned}$$

Since ξ is a density point for all $\varepsilon > 0$, there exists n sufficiently large such that

$$\frac{\nu_x(\mathcal{S}_R(x,\gamma_n x)\cap A^c)}{\nu_x(\mathcal{S}_R(x,\gamma_n x))} \leq \varepsilon.$$

It follows that

$$\frac{\nu_x(\mathcal{S}_R(\gamma_n^{-1}x,x)\cap A)}{\nu_x(\mathcal{S}_R(\gamma_n^{-1}x,x))} \geq 1 - K\varepsilon. \tag{5}$$

Finally, from Lemma 4.14 there exists R sufficiently large such that for all γ_n

$$\nu_x(\mathcal{S}_R(\gamma_n^{-1}x,x)) \geq \nu_x(\Lambda) - \varepsilon.$$

Putting everything together we have

$$\begin{aligned}
\nu_x(A) &\geq \nu_x(\mathcal{S}_R(\gamma_n^{-1}x,x)\cap A) \\
&\geq (1-K\varepsilon)\nu_x(\mathcal{S}_R(\gamma_n^{-1}x,x)) \quad \text{From } (Eq.5) \\
&\geq (1-K\varepsilon)(\nu_x(\Lambda)-\varepsilon).
\end{aligned}$$

Since ε is arbitrary, we have $\nu_x(A) = \nu_x(\Lambda)$, this shows the ergodicity of ν. ☐

Proposition 4.16. *The Patterson-Sullivan density is the only conformal density up to a multiplicative constant.*

Proof. Let ν be the Patterson-Sullivan density, and let ν' be another conformal density. We know from Corollary 4.12, we know that the dimension of ν' is δ_Γ. It follows that $\nu+\nu'$ is also a conformal density of dimension δ. The measure ν_x is absolutely continous with respect to $\nu_x+\nu'_x$, so they differ by a density function. Since these measures are both conformal measures, this function is invariant under Γ. It follows from the ergodicity that this function is constant $\nu_x+\nu'_x$-almost everywhere (the constant being independant of x), so ν and $\nu+\nu'$ are proportional, hence the proportionality between ν and ν'. □

5 Lorentzian Hausdorff dimension and measure

We introduce the concept of Lorentzian Hausdorff dimension, using the usual definition in metric space to our case. This gives an invariant that we show to be equal to the critical exponent, Theorem 5.3. Moreover using a comparison with a Riemannian metric we show an inequality in every dimension : $\delta(\Gamma) \leq n-1$.

5.1 General definitions

The Hausdorff dimension of a metric space reflects the number of balls of a certain radius that are necessary to cover the set. In Lorentzian geometry, we find an analogue notion, replacing Riemannian balls by hyperboloids.

Although we could define a notion of Lorentzian Hausdorff dimension and measures in any Lorentzian manifold, dealing with this general setting would be the source of many technical difficulties. Since the limit set of a quasi-Fuchsian group of $\mathrm{SO}(2,n)$ is included in the de Sitter space, it is enough for our purpose to deal with subsets of dS^n.

If $A \subset \mathrm{dS}^n$, we set $\mathrm{H}^{s,\varepsilon}_{\mathrm{dS}}(A) = \inf\{\sum r_i^s | A \subset \bigcup B_{\mathrm{dS}}(x_i,r_i), x_i \in A, r_i \leq \varepsilon\}$, and $\mathrm{H}^s_{\mathrm{dS}}(A) = \lim_{\varepsilon\to 0} \mathrm{H}^{s,\varepsilon}_{\mathrm{dS}}(A) \in [0\,,+\infty]$. Finally, the Lorentzian Hausdorff dimension of A is $\mathrm{Hdim}_{\mathrm{dS}}(A) = \inf\{s | \mathrm{H}^s_{\mathrm{dS}}(A) = 0\}$.

Proposition 5.1. *If* $A \subset \mathrm{dS}^n$*, then* $\mathrm{Hdim}_{\mathrm{dS}}(A) \leq \mathrm{Hdim}_h(A)$*, where* $\mathrm{Hdim}_h(A)$ *is the Hausdorff dimension with respect to any Riemannian metric* h*.*

Proof. First, we see that there is a Riemannian metric h on dS^n such that $d_h(\xi,\eta) \geq \theta(\xi,\eta)$ for all $\xi,\eta \in \mathrm{dS}^n$. For this, we use the hyperboloid model $\mathrm{dS}^n = \{(x_0,\dots,x_n) \in \mathbb{R}^{n+1} \mid -x_0^2 + x_1^2 + \cdots + x_n^2 = 1\}$. If $\xi,\eta \in \mathrm{dS}^n$, then $\theta(\xi,\eta) > 0$ is equivalent to $\langle\xi|\eta\rangle_{1,n} < 1$, and in this case we have $\theta(\xi,\eta) = \arccos(\langle\xi|\eta\rangle_{1,n}) = \arccos(\frac{1-\langle\xi-\eta|\xi-\eta\rangle_{1,n}}{2}) \leq \sqrt{|\langle\xi-\eta|\xi-\eta\rangle_{1,n}|}$.
If we denote by $\langle\cdot|\cdot\rangle_E$ the Euclidean inner product on $\mathbb{R}^{n+1}$, we find that $\sqrt{|\langle\xi-\eta|\xi-\eta\rangle_{1,n}|} \leq \sqrt{\langle\xi-\eta|\xi-\eta\rangle_E}$ for all $\xi,\eta \in \mathrm{dS}^n$ (because $\langle\xi-\eta|\xi-\eta\rangle_{1,n} \geq 0$), hence $\theta(\xi,\eta) \leq \sqrt{\langle\xi-\eta|\xi-\eta\rangle_E}$. If h is the Riemannian metric

on dS^n induced by the Euclidean metric of $\mathbb{R}^{n+1}$, we finally get $\theta(\xi,\eta) \leq d_h(\xi,\eta)$.
From this, we deduce that $B(\xi,r) \subset B_h(\xi,r)$ for all $\xi \in \mathrm{dS}^n, r \geq 0$, where B_h is the ball for the Riemannian metric h. From this we deduce that $\dim_{\mathrm{LH}}(A)$ is smaller than the Hausdorff dimension for h, and the Hausdorff dimension does not depend on the choice of a Riemannian metric. □

5.2 The Lorentzian Hausdorff dimension of the limit set

Lemma 5.2 (Vitali for d_x)**.** *Given a subset $J \subset \Lambda$ and a bounded function $r : J \to (0,+\infty)$, there is a subset $I \subset J$ such that:*

- *The balls $B_x(\xi, r(\xi))$ are disjoint for distinct points $\xi \in I$.*
- $\bigcup_{\xi \in J} B_x(\xi, r(\xi)) \subset \bigcup_{\eta \in I} B_x(\eta, 5\lambda^2 r(\eta))$.

Proof. Let $R = \sup_J r$, and consider $J_n = \{\xi \in J : 2^{-n-1}R < r(\xi) \leq 2^{-n}R\}$ for any $n \geq 0$, so that J is the disjoint unions of these subsets. Define inductively subsets I_n, H_n of J_n by letting $h_0 = J_0$, and $I_0 \subset F_0$ be maximal amongst the subsets $A \subset H_0$ such that the balls $B_x(\xi, r(\xi))$ are disjoint for distinct $\xi \in A$ (such a subset exists by Zorn's Lemma). Given $I_0, \ldots, I_n$, we let $H_{n+1} = \{\xi \in F_n : \forall \eta \in I_0 \cup \cdots \cup I_n \ B_x(\xi, r(\xi)) \cap B_x(\eta, r(\eta)) = \emptyset\}$, and choose G_{n+1} maximal $A \subset H_{n+1}$ such that the balls $B_x(\xi, r(\xi))$ are disjoint for distinct $\xi \in A$. Finally, let $I = \bigcup_{n \in \mathbb{N}} I_n$.
It follows from the construction of I that the considered balls are disjoint.
For the second point, let $\xi \in J$. For this purpose, consider $n \in \mathbb{N}$ such that $\xi \in J_n$. There are two cases: either $\xi \notin H_n$, in which case there is $\eta \in I_0 \cup \cdots \cup I_n \subset I$ satisfying $B_x(\xi, r(\xi)) \cap B_x(\eta, r(\eta)) \neq \emptyset$, or $\xi \in H_n$, in which case there is $\eta \in I_n \subset I$ satisfying $B_x(\xi, r(\xi)) \cap B_x(\eta, r(\eta)) \neq \emptyset$ (because of the maximality of I_n).
In both cases, we find $\eta \in I_0 \cup \cdots \cup I_n$ such that $B_x(\xi, r(\xi)) \cap B_x(\eta, r(\eta)) \neq \emptyset$. Since $r(\eta) > 2^{-n-1}R$ and $r(\xi) \leq 2^{-n}R$, we have $r(\xi) \leq 2r(\eta)$, which implies $B_x(\xi, r(\xi)) \subset B_x(\eta, \lambda(2+3\lambda)r(\eta)) \subset B_x(\eta, 5\lambda^2 r(\eta))$.
□

Theorem 5.3. $\delta_\Gamma = \mathrm{Hdim}_{\mathrm{dS}}(\Lambda)$

Instead of working directly with the Lorentzian Hausdorff measures, we will deal with the Hausdorff measures associated to d_x, and use the same notations $\mathrm{H}^{s,\varepsilon}_{d_x}$, $\mathrm{H}^s_{d_x}$ and Hdim_{d_x}.
Since $d_x(\xi,\eta) = \sin\frac{\theta_x(\xi,\eta)}{2}$, we find that $\mathrm{H}^{s,\varepsilon}_{d_x}(A) = 2^{-s}\mathrm{H}^{s,\varepsilon}_{\mathrm{dS}}(A)$ for any measurable set A, and $\mathrm{Hdim}_{d_x}(A) = \mathrm{Hdim}_{\mathrm{dS}}(A)$.

We prove successively the two inequalities.

Proposition 5.4. $\delta_\Gamma \geq \mathrm{Hdim}_{\mathrm{dS}}(\Lambda)$

Proof. By Theorem 4.9, there is a constant $c > 0$ such that $r^{\delta_\Gamma} \leq c\nu_x(B_x(\xi, r))$ for all $\xi \in \Lambda$, $r \in (0, 1)$.
Let $\varepsilon > 0$, and let $\xi_1, \ldots, \xi_p$ be a finite set such that $\Lambda \subset \bigcup_{1 \leq i \leq p} B_x(\xi_i, \frac{\varepsilon}{5\lambda^2})$.
By Lemma 5.2, we can find a subset $J \subset \{1, \ldots, p\}$ such that $\Lambda \subset \bigcup_{i \in J} B_x(\xi_i, \varepsilon)$ and the balls $B_x(\xi_i, \frac{\varepsilon}{5\lambda^2})$ for $i \in J$ are pairwise disjoint.
It follows that $\mathrm{H}_{d_x}^{\delta_\Gamma, \varepsilon}(\Lambda) \leq \sum_{i \in J} \varepsilon^{\delta_\Gamma}$, hence:

$$
\begin{aligned}
\mathrm{H}_{d_x}^{\delta_\Gamma, \varepsilon}(\Lambda) &\leq (5\lambda^2)^{\delta_\Gamma} c \sum_{i \in J} \nu_x(B_x(\xi_i, \frac{\varepsilon}{5\lambda^2})) \\
&\leq (5\lambda^2)^{\delta_\Gamma} c\nu_x(\Lambda)
\end{aligned}
$$

This shows that $\mathrm{H}_{d_x}^{\delta_\Gamma}(\Lambda) < +\infty$, and $\mathrm{Hdim}_{\mathrm{dS}}(\Lambda) = \mathrm{Hdim}_{d_x}(\Lambda) \leq \delta_\Gamma$. □

Proposition 5.5. $\delta_\Gamma \leq \mathrm{Hdim}_{\mathrm{dS}}(\Lambda)$

Proof. Let (ξ_i, r_i) be a countable family of points of Λ and radii such that $\Lambda \subset \bigcup B_x(\xi_i, r_i)$. By Theorem 4.9, there is a constant $c > 0$ such that $\nu_x(B_x(\xi, r)) \leq c r^{\delta_\Gamma}$ for all $\xi \in \Lambda$, $r \in (0, 1)$. It follows that $\nu_x(\Lambda) \leq \sum \nu_x(B_x(\xi_i, r_i)) \leq c \sum r_i^{\delta_\Gamma}$, hence $\mathrm{H}_{d_x}^{\delta_\Gamma}(\Lambda) \geq \frac{\nu_x(\Lambda)}{c} > 0$, and $\mathrm{Hdim}_{\mathrm{dS}}(\Lambda) = \mathrm{Hdim}_{d_x}(\Lambda) \geq \delta_\Gamma$. □

We obtain the announced inequality

Corollary 5.6. $\delta_\Gamma \leq n - 1$

5.3 Lorentzian Hausdorff measure and the Patterson-Sullivan density

The estimates on the Hausdorff measure of Λ can actually be carried to any subset of Λ, hence giving absolute continuity between the Hausdorff measure and the Patterson-Sullivan measure.

Theorem 5.7. *Let ν denote the Patterson-Sullivan density. For all $x \in C(\Lambda)$, there is $\alpha > 0$ such that $\frac{1}{\alpha}\mathrm{H}_{\mathrm{dS}}^{\delta_\Gamma}(E) \leq \nu_x(E) \leq \alpha \mathrm{H}_{\mathrm{dS}}^{\delta_\Gamma}(E)$ for all measurable subset $E \subset \Lambda$.*

Proof. Let $E \subset \Lambda$ be a measurable set.
We start with the left hand side inequality. Let $\varepsilon > 0$. Consider the open cover $E \subset \bigcup_{\xi \in E} B_x(\xi, \frac{\varepsilon}{5\lambda^2})$. By Lemma 5.2, we can find a (necessarily countable) subset $J \subset E$ such that $E \subset \bigcup_{\xi \in J} B_x(\xi, \varepsilon)$ and the balls $B_x(\xi, \frac{\varepsilon}{5\lambda^2})$ for $\xi \in J$ are pairwise disjoint.
By Theorem 4.9, there is a constant $c > 0$ such that $c\nu_x(B_x(\xi, r)) \geq r^{\delta_\Gamma}$ for all $\xi \in \Lambda$, $r \in (0, 1)$. Since $\mathrm{H}_{d_x}^{\delta_\Gamma, \varepsilon}(E) \leq \sum_{\xi \in J} \varepsilon^{\delta_\Gamma}$, we find:

$$
\begin{aligned}
\mathrm{H}_{d_x}^{\delta_\Gamma, \varepsilon}(E) &\leq (5\lambda^2)^{\delta_\Gamma} c \sum_{\xi \in J} \nu_x(B_x(\xi, \frac{\varepsilon}{5\lambda^2})) \\
&\leq (5\lambda^2)^{\delta_\Gamma} c\nu_x(E)
\end{aligned}
$$

Let us now deal with the right hand side inequality. Let (ξ_i, r_i) be a countable family of points of E and radii such that $E \subset \bigcup B_x(\xi_i, r_i)$. By Theorem 4.9, there is a constant $c > 0$ such that $\nu_x(B_x(\xi, r)) \leq cr^{\delta_\Gamma}$ for all $\xi \in \Lambda$, $r \in (0, 1)$. It follows that $\nu_x(E) \leq \sum \nu_x(B_x(\xi_i, r_i)) \leq c \sum r_i^{\delta_\Gamma}$, hence $\mathrm{H}^{\delta_\Gamma}_{d_x}(E) \geq \frac{\nu_x(E)}{c}$.
Combining these two inequalities, we get:

$$\frac{1}{c(5\lambda^2)^{\delta_\Gamma}} \mathrm{H}^{\delta_\Gamma}_{d_x}(E) \leq \nu_x(E) \leq c\mathrm{H}^{\delta_\Gamma}_{d_x}(E)$$

□

6 Rigidity theorem in dimension 3

In this section we prove a rigidity theorem for the critical exponent in dimension 3. The proof is based on a comparison of the critical exponent for Γ acting on the convex core and on its boundary. The proof is quite similar to the Hyperbolic case where it can be found in [Glo15b]. We will need the equivalent of the Bowen-Margulis measure on the non-wandering set for the geodesic flow on $C(\Lambda)$. We explain the construction and proves the ergodicity of this measure in the first part, then comes the genuine proof of rigidity.

As we said in the introduction, thanks to Mess parametrization of AdS quasi-Fuchsian groups, this theorem is equivalent to a Theorem of C. Bishop and T. Steger [BS91] for pair of Fuchsian representations acting on $\mathbb{H}^2 \times \mathbb{H}^2$. The proof proposed here is totally independent and has a clear geometric interpretation.

Moreover thanks to examples proposed in [Glo15a], we know the asymptotic behaviour of critical exponent when the two representations in Mess parametrizations range over the product of Teichmüller spaces.

6.1 Bowen - Margulis measure

This measure has been first introduced by G. Margulis in [Mar69] and R. Bowen in [Bow72]. A good introduction for the hyperbolic case can be found in Chapter 8 of Nicholls' book [Nic89] or in the book of T. Roblin [Rob03].

The geodesic current $d\mu(\xi, \eta) := \frac{d\nu_x(\xi) d\nu_x(\eta)}{d_x(\xi,\eta)^{-2\delta(\Gamma)}}$ does not depend on the point x. Indeed,

$$\frac{d\nu_x(\xi) d\nu_x(\eta)}{d\nu_y(\xi) d\nu_y(\eta)} = e^{-\delta(\beta_\xi(x,y) + \beta_\eta(x,y))}$$

And

$$\frac{d_x(\xi, \eta)^2}{d_y(\xi, \eta)^2} = \frac{-\langle \xi | \eta \rangle}{\langle \xi | x \rangle \langle x | \eta \rangle} \frac{\langle \xi | y \rangle \langle y | \eta \rangle}{-\langle \xi | \eta \rangle} = \frac{\langle \xi | y \rangle \langle y | \eta \rangle}{\langle \xi | x \rangle \langle x | \eta \rangle}$$

Finally from the definition of β we have :

$$\frac{\langle \xi \,|\, y\rangle}{\langle \xi \,|\, x\rangle} = e^{-\beta_\xi(x,y)}.$$

$$\frac{\langle \eta \,|\, y\rangle}{\langle \eta \,|\, x\rangle} = e^{-\beta_\eta(x,y)}.$$

Hence

$$\frac{d_x(\xi,\eta)^{2\delta}}{d_y(\xi,\eta)^{2\delta}} = e^{-\delta(\beta_\xi(x,y)+\beta_\eta(x,y))}.$$

The same kind of computations shows that μ is also Γ-invariant.

We denote by v^+ respectively v^- the limit of the geodesic ray generated by v resp. $-v$

Definition 6.1. *The non-wandering set, denoted by $N^1C(\Lambda)$, is the subset of $T^1C(\Lambda)$ defined by*

$$N^1C(\Lambda) := \{v \in T^1C(\Lambda) \,|\, v^\pm \in \Lambda\}.$$

This is homomorphic to $\Lambda^{(2)} \times \mathbb{R}$. The action of the geodesic flow ϕ_t is given by

$$\phi_t(v) = \phi_t(v^-, v^+, s) = (v^-, v^+, t+s).$$

Thanks to previous computations, we see that $N^1C(\Lambda)$ carries a measure, invariant by Γ as well as by ϕ_t.

Definition 6.2. *The following measure on $N^1C(\Lambda) \simeq \Lambda^{(2)} \times \mathbb{R}$ is called the* Bowen-Margulis *measure :*

$$dm(v) := \frac{d\nu_x(v^-)d\nu_x(^+)dt}{d_x(v^-, v^+)^{-2\delta(\Gamma)}}.$$

It is invariant by Γ and ϕ_t.

In other words, let $f : N^1(C(\Lambda)) \to \mathbb{R}$ be a continuous function with compact support, and let $t \to c_{(\xi\eta)}(t)$ be a parametrization of the AdS geodesic $(\xi\eta)$. Then

$$\int_{N^1C(\Lambda)} f(v)dm(v) := \int_{\Lambda^{(2)}} \int_{\mathbb{R}} f(c_{(\xi\eta)}(t))dtd\mu(\xi,\eta).$$

We easily see in this formulation that it is invariant by the geodesic flow.

Remark that $N^1C(\Lambda)$ is invariant by Γ, and as a closed subset in $T^1C(\Lambda)$, $N^1C(\Lambda)/\Gamma$ is compact. The Bowen Margulis goes down to a ϕ_t-invariant finite measure whose support is $N^1C(\Lambda)/\Gamma$, still denoted by m.

Theorem 6.3. *The Bowen Margulis measure is ergodic.*

Proof. Since m is already a locally product measure, this is a straightforward consequence of the so-called Hopf's argument. Let us recall the proof. Let f be a continuous function on $N^1C(\Lambda)$. We consider the Birkhoff means :

$$\Phi_f^s(v) := \lim_{T\to+\infty} \frac{1}{T}\int_0^T f(\phi_t v)dt.$$

$$\Phi_f^u(v) := \lim_{T\to+\infty} \frac{1}{T}\int_0^T f(\phi_{-t} v)dt.$$

Thanks to Birkhoff ergodic theorem the functions Φ_f^u and Φ_f^s exist and are equal almost everywhere. Clearly, Φ_f^u and Φ_f^s are invariant by the geodesic flow. Moreover Φ_f^u is constant along unstable leaves of $N^1C(\Lambda)$ that is $W^u(v) := \{w = (v^-, x, t) \in N^1C(\Lambda) \,|\, x \in \Lambda\backslash\{v^+\}\}$ and Φ_f^s is constant along the stable leaves of $N^1C(\Lambda)$ that is $W^s(v) := \{w = (x, v^+, t) \in N^1C(\Lambda) \,|\, x \in \Lambda \setminus \{v^-\}\}$.

Now, since the measure is a product measure, it is a consequence of Fubini that Φ_f^u and Φ_f^s are almost everywhere constant. This concludes the proof.

□

6.2 Proof of inequality in Theorem 6.5

Remark concerning notations In this section we will have to make a clear distinction between the different metrics, therefore we will add a subscript to the distance defined in 3.1 and write d_{AdS} instead of d.
Let $\Gamma \subset \mathrm{SO}(2,2)$ be a AdS^3 quasi-Fuchsian group. The intrinsic distance on a complete Γ-invariant Riemannian hypersurface $\widetilde{\Sigma}$ will be denoted by $d_{\widetilde{\Sigma}}$. The balls for d_{AdS} will be denoted by $B_{\mathrm{AdS}}(x, R) \subset C(\Lambda)$, the balls for $d_{\widetilde{\Sigma}}$ will be denoted by $B_{\widetilde{\Sigma}}(x, R) \subset \widetilde{\Sigma}$. The Lorentzian shadows for the metric d_{Ads} will be denoted by $\mathcal{S}_{\mathrm{AdS}}(x, y, R)$, the Riemannian shadows for the metric $d_{\widetilde{\Sigma}}$ will be denoted by $\mathcal{S}_{\widetilde{\Sigma}}(x, y, R)$.

To avoid multiplication of indices, we will write AdS instead of AdS^3 in all this section. We don't use the dimension in the majority of the arguments, in fact it is only used in two results. It is first used in the existence of a Cauchy surface of entropy 1, (namely the boundary of the convex core). The existence in higher dimension of a Cauchy surface of entropy $n-1$ is not known. The second times we use the dimension is when we computed and compare length spectrum between $\widetilde{\Sigma}$ and $C(\Lambda)/\Gamma$. This argument can in fact be bypassed. Therefore, the rigidity in higher dimension is equivalent to the existence of a Cauchy surface of entropy $n-1$.

Let $\widetilde{\Sigma}$ be a negatively curved, Cauchy surface, Γ invariant embedded disk in $C(\Lambda)$. We allow $\widetilde{\Sigma}$ to be bend along geodesic lamination. It includes two basic examples: the boundaries of the convex core which are isometric to $\mathbb{H}^2$, bend along laminations and the unique maximal surface.

Let V be a time-like vector field in $E(\Lambda)/\Gamma$, we still call V its lift to $E(\Lambda)$. For any x in $C(\Lambda)$ we call $f(x)$ the intersection of the integral curve of V starting at x and $\widetilde{\Sigma}$. This is well defined since $\widetilde{\Sigma}$ is a Cauchy surface.

Let (ϕ_t) be the geodesic flow on $T^1C(\Lambda)$. We may write ϕ_t^{AdS} if we want to stress that it is the geodesic flow on AdS and similarly $\phi_t^{\widetilde{\Sigma}}$ for the geodesic flow on $\widetilde{\Sigma}$. Let $\pi : T^1C(\Lambda) \to C(\Lambda)$ denote the projection. For any $v \in T^1C(\Lambda)$ we define the following cocycle: $a(v,t) = d_{\widetilde{\Sigma}}(f(\pi\phi_t(v)), f(\pi v))$. It is subadditive:

$$\begin{aligned} a(v, t_1 + t_2) &= d_{\widetilde{\Sigma}}(f(\pi\phi_{t_1+t_2}(v)), f(\pi v)) \\ &\leq d_{\widetilde{\Sigma}}(f(\pi\phi_{t_1+t_2}(v)), f(\pi\phi_{t_1}v)) + d_{\widetilde{\Sigma}}(f(\pi\phi_{t_1}(v)), f(\pi v)) \\ &\leq a(\phi_{t_1}v, t_2) + a(v, t_1) \end{aligned}$$

Since a is Γ invariant it defines a subadditive cocycle on $N^1C(\Lambda)/\Gamma$, still denoted by a.

The following is a consequence of Kingman's subadditive ergodic theorem

Theorem 6.4. *[Kin68] Les μ be a ϕ_t invariant probability measure on $N^1C(\Lambda)/\Gamma$. Then*

$$I_\mu(\Sigma, v) := \lim_{t\to\infty} \frac{a(v,t)}{t}$$

exists for μ almost $v \in N^1C(\Lambda)/\Gamma$ and defines a μ-integrable function on $N^1C(\Lambda)/\Gamma$, invariant under the geodesic flow and we have :

$$\int_{N^1C(\Lambda)/\Gamma} I_\mu(\Sigma, v) d\mu = \lim_{t\to\infty} \int_{N^1C(\Lambda)/\Gamma} \frac{a(v,t)}{t} d\mu.$$

Moreover if μ is ergodic $I_\mu(\Sigma, v)$ is constant μ-almost everywhere. In this case, we write $I_\mu(\Sigma)$

We will prove the following theorem which is a stronger form of the rigidity theorem.

Theorem 6.5. *Let $h(\Sigma)$ be the volume entropy of $\widetilde{\Sigma}$. Let $\delta(\Gamma)$ be the critical exponent of Γ on AdS^3. Finally let m be the Bowen-Margulis measure on $N^1C(\Lambda)/\Gamma$. Then*

$$h(\Sigma) I_m(\Sigma) \geq \delta(\Gamma), \tag{6}$$

with equality if and only if the marked length spectra of Σ and M are proportional and the proportionality constant is given by $I_m(\Sigma)$.

We obtain as a corollary the following rigidity theorem:

Corollary 6.6. $\delta(\Gamma) \leq 1$ *with equality if and only if Γ is Fuchsian.*

In order to compare the distance on the $\widetilde{\Sigma}$ and AdS we will need the following proposition.

Proposition 6.7. *Let Ω be a Γ-invariant open bounded convex set that contains $C(\Lambda)$, and denote by d_H the Hilbert distance of Ω. There is a constant $L > 0$ such that $\frac{1}{L}d_H(x,y) \leq d_{\mathrm{AdS}}(x,y) \leq Ld_H(x,y)$ for all $x, y \in \partial_+C(\Lambda)$, where $\partial_+C(\Lambda)$ is the future boundary of the convex core.*

Proof. Recall that a Hilbert metric is a Finsler metric, where the geodesic are affine lines [Cra11]. Denote by N the Finsler norm on $T\Omega$ associated to the Hilbert metric.
Given $x \in C(\Lambda)$ and $v \in T_x\mathrm{AdS}^{n+1}$, we denote by $v_\pm \in \partial C(\Lambda)$ the intersections of $\partial C(\Lambda) \subset \overline{\mathrm{AdS}^{n+1}}$ with the geodesic generated by v. Remark that, from the definition, $v_\pm$ does not necessary belong to the boundary of AdS.
Let $V = \{v \in T\mathrm{AdS}^{n+1} : N(v) = 1, v_\pm \in \partial_+C(\Lambda) \cup \Lambda\}$.
First, notice that V is Γ-invariant. Let us show that the action of Γ on V is cocompact. Let $K \subset C(\Lambda)$ be a compact set such that $\Gamma.K = C(\Lambda)$, and let $V_K = \{v \in V : v \in T_x\mathrm{AdS}^{n+1}, x \in K\}$. Since the maps $v_\pm$ are continuous, it follows that V_K is a closed subset of the unit tangent bundle over K, therefore is compact. Since $\Gamma.V_K = V$, it follows that the action of Γ on V is cocompact.
As a consequence, the Γ-invariant function $\langle v|v\rangle$ is bounded, and bounded away from 0 (any vector in V is spacelike because $\partial_+C(\Lambda)$ is a Cauchy hypersurface): let $L > 0$ be such that $\frac{1}{L^2} \leq \langle v|v\rangle \leq L^2$ for all $v \in V$.
Let $x, y \in \widetilde{\Sigma}$. Denote by $x(t)$ the geodesic going from x to y, so that $d_{\mathrm{AdS}}(x,y) = \int_0^1 \sqrt{\langle \dot{x}(t)|\dot{x}(t)\rangle}dt$ and $d_H(x,y) = \int_0^1 N(\dot{x}(t))dt$. The tangent vectors $\dot{x}(t)$ are multiples of vectors of V, hence $\frac{N(\dot{x}(t))}{L} \leq \sqrt{\langle \dot{x}(t)|\dot{x}(t)\rangle} \leq LN(\dot{x}(t))$. Integrating this inequality yields the proposition.

□

Examples of such sets Ω are ε-neighbourhoods of $C(\Lambda)$ (given $\varepsilon > 0$, it is the set of points that can be joined by a timelike geodesic of length less than ε to a point of $C(\Lambda)$), or the whole black domain (except in the Fuchsian case).

Proposition 6.8. *The function $f : C(\Lambda) \to \widetilde{\Sigma}$ is a quasi-isometry, quasi 1-Lipschitz ie : $\exists K_1 > 1$, $K_2 > 0$, $\forall x, y \in C(\Lambda)$:*

$$\frac{1}{K_1}d_{\mathrm{AdS}}(x,y) - K_1 \leq d_{\widetilde{\Sigma}}(f(x), f(y)) \leq d_{\mathrm{AdS}}(x,y) + K_2.$$

Proof. First remark that we can suppose that $x, y \in \widetilde{\Sigma}$. Indeed if $x, y \in C(\Lambda)$, since $d_{\mathrm{AdS}}(x, f(x)) = d_{\mathrm{AdS}}(y, f(y)) = 0$ then $d_{\mathrm{AdS}}(x,y) - 2k \leq d_{\mathrm{AdS}}(f(x), f(y)) \leq d_{\mathrm{AdS}}(x,y) + 2k$ where $k > 0$ is given by the triangle inequality 3.4.

Let us prove the left inequality. The group Γ acts properly discontinuously and cocompactly on $\widetilde{\Sigma}$ as well as on the neighbourhood of $C(\Lambda)$ as defined in Proposition 6.7. Hence by Svarc-Milnor lemma, $\widetilde{\Sigma}$ is quasi-isometric to this neighbourhood of $C(\Lambda)$, which is by Proposition 6.7 Lipschitz equivalent at the AdS distance.

We now prove the right inequality. If $x, y \in \widetilde{\Sigma}$, take any plane P of signature $(+,-)$ containing (x,y). In an affine chart, this plane is isometric to $\left(\mathbb{R} \times (-\pi/2, \pi/2), dt^2 - \cosh^2(t)d\theta\right)$, where dt^2 is the Lorentzian distance on the space like geodesic (x,y). Now since $\widetilde{\Sigma}$ is a Cauchy surface, the intersection of P with $\widetilde{\Sigma}$ is a graph over $\mathbb{R}$, in particular the length $\ell = \int_0^{d_{\mathrm{AdS}}(x,y)} \sqrt{1-\cosh^2(t)\theta'(t)}dt \leq d_{\mathrm{AdS}}(x,y)$ of the curve from x to y in $P \cap \widetilde{\Sigma}$ is smaller than $d_{\mathrm{AdS}}(x,y)$. The $\widetilde{\Sigma}$-distance between x and y is smaller than ℓ (since $\widetilde{\Sigma}$ is Riemannian) hence smaller than $d_{\mathrm{AdS}}(x,y)$.

□

We give a series of simple corollaries that we will use during the proof of Theorem 6.5.

Corollary 6.9. *For every ergodic μ,*

$$I_\mu(\Sigma) \leq 1$$

Proof. Let v be a typical vector for μ we have

$$\begin{aligned} \frac{a(v,t)}{t} &= \frac{d_{\widetilde{\Sigma}}(f(\pi\phi_t(v)), f(\pi v))}{d_{\mathrm{AdS}}(\pi\phi_t(v), \pi v)} \\ &\leq 1 + \frac{K}{t} \end{aligned}$$

taking the limit concludes the proof. □

Corollary 6.10. *There exists $K > 0$ such that, for all $p \in C(\Lambda)$ and all $\xi \in \Lambda$, then $f([p,\xi))$ is at distance ($d_{\widetilde{\Sigma}}$) at most K of the unique geodesic on $\widetilde{\Sigma}$ from p to ξ.*

Proof. Let $\xi \in \Lambda$ and $v_p(\xi)$ be the unit-space like vector in $T^1_pC(\Lambda)$ such that $\lim_{t\to\infty} \phi_t^{\mathrm{AdS}}(v_p(\xi)) = \xi$. Consider the curve $c : \mathbb{R}^+ \to \widetilde{\Sigma}$ defined by $c(t) := f(\pi\phi_t^{\mathrm{AdS}}(v_p(\xi))$. Then from Proposition 6.8, for all $t, s \in \mathbb{R}^+$ we have

$$\frac{|t-s|}{K_1} - K_1 \leq d_{\widetilde{\Sigma}}(c(t), c(s)) \leq |t-s| + K_2.$$

Hence c is a quasi-geodesic. Thanks to Morse Lemma, since by hypothesis $\widetilde{\Sigma}$ is negatively curved, the quasi-geodesic c is at distance ($d_{\widetilde{\Sigma}}$) at most K of a unique geodesic [Bal85]. This constant K depends only on $\widetilde{\Sigma}$ and K_1, K_2, in particular it does not depends on p and ξ. □

Corollary 6.11. *For all R there exists R' such that for all $x \in C(\Lambda)$:*

$$B_{\mathrm{AdS}}(x,R) \cap \widetilde{\Sigma} \subset B_{\widetilde{\Sigma}}(f(x),R').$$

For all R there exists R' such that for all $x \in \widetilde{\Sigma}$:

$$B_{\widetilde{\Sigma}}(x,R) \subset B_{\mathrm{AdS}}(x,R').$$

Proof. We prove the first inclusion. Let $y \in B_{\mathrm{AdS}}(x,R) \cap \widetilde{\Sigma}$.

$$\begin{aligned} d_{\widetilde{\Sigma}}(f(x),y) &= d_{\widetilde{\Sigma}}(f(x),f(y)) \\ &\leq d_{\mathrm{AdS}}(x,y)+K_1 \\ &= R+K_1 \end{aligned}$$

And we can choose any R' larger than $R+K_1$.

We prove the second inclusion. Let $y \in B_{\widetilde{\Sigma}}(x,R)$ then

$$\begin{aligned} d_{\mathrm{AdS}}(x,y) &\leq K_1 d_{\widetilde{\Sigma}}(f(x),f(y))+K_1^2 \\ &\leq k_1 R+K_1^2. \end{aligned}$$

And we can choose any R' larger than $K_1R+K_1^2$. □

We fix a point $p \in C(\Lambda)$. Shadows in AdS are supposed to be centred in p, shadows in $\widetilde{\Sigma}$ are supposed to be centred in $f(p)$.

Corollary 6.12. *For all R there exists R'' such that for all $x \in C(\Lambda)$*

$$\mathcal{S}_{\mathrm{AdS}}(x,R) \subset \mathcal{S}_{\widetilde{\Sigma}}(f(x),R'').$$

For all R there exists R'' such that for all $x \in \widetilde{\Sigma}$

$$\mathcal{S}_{\widetilde{\Sigma}}(x,R) \subset \mathcal{S}_{\mathrm{AdS}}(x,R'').$$

Proof. We prove the first inclusion. Let ξ be in $\mathcal{S}_{\mathrm{AdS}}(x,R)$, then by definition the AdS geodesic $[p,\xi)_{\mathrm{AdS}}$ intersects $B_{\mathrm{AdS}}(x,R)$. By Corollary 6.11, this implies that

$$f([p,\xi)_{\mathrm{AdS}}) \cap B_{\widetilde{\Sigma}}(f(x),R') \neq \emptyset.$$

Moreover by Corollary 6.10 this implies that the unique geodesic from $f(p)$ to ξ intersects $B_{\widetilde{\Sigma}}(f(x),R'')$. This means by definition that $\xi \in \mathcal{S}_{\widetilde{\Sigma}}(f(x),R'')$.

We prove the second inclusion. Let ξ be in $\mathcal{S}_{\widetilde{\Sigma}}(x,R)$, then by definition the $\widetilde{\Sigma}$ geodesic $[p,\xi)_{\widetilde{\Sigma}}$ intersects $B_{\widetilde{\Sigma}}(x,R)$. Let $[p,\xi)_{\mathrm{AdS}}$ be the AdS geodesic, then by Corollary 6.10 $f([p,\xi)_{\mathrm{AdS}})$ is at bounded distance of $[p,\xi)_{\widetilde{\Sigma}}$, let $z \in [p,\xi)_{\mathrm{AdS}}$ such that $d_{\widetilde{\Sigma}}(f(z),x) \leq R+K$. We then have

$$\begin{aligned} d_{\mathrm{AdS}}(z,x) &\leq K_1 d_{\widetilde{\Sigma}}(f(z),f(x))+K_1^2 \\ &\leq K_1(R+K)+K_1^2 \end{aligned}$$

Then we can choose any R'' larger than $K_1(R+K)+K_1^2$. □

Corollary 6.13. *There exists* $K > 0$ *such that for every* $v \in T^1\mathrm{AdS}$, $t_1, t_2 > 0$ *we have*

$$a(v,t_1) + a(\phi_{t_1}^{\mathrm{AdS}}(v), t_2) \le a(v, t_1+t_2) + K$$

Proof. This is a consequence of 6.10. Let z_1 be a point on the geodesic $[f(p),\xi)_{\widetilde{\Sigma}}$ such that $d_{\widetilde{\Sigma}}(z_1, f(\pi\phi_{t_1}^{\mathrm{AdS}}(v)) \le K$. Let z_2 be a point on the geodesic $[f(p),\xi)_{\widetilde{\Sigma}}$ such that $d_{\widetilde{\Sigma}}(z_2, f(\pi\phi_{t_1+t_2}^{\mathrm{AdS}}(v))) \le K$. Then

$$a(v,t_1) = d_{\widetilde{\Sigma}}(f(p), f(\pi\phi_{t_1}^{\mathrm{AdS}}(v))) \le d_{\widetilde{\Sigma}}(f(p), z_1) + K$$

and

$$a(\phi_{t_1}^{\mathrm{AdS}}(v), t_2) = d_{\widetilde{\Sigma}}(f(\pi\phi_{t_1}^{\mathrm{AdS}}(v)), f(\pi\phi_{t_1+t_2}^{\mathrm{AdS}}(v))) \le d_{\widetilde{\Sigma}}(z_1, z_2) + 2K.$$

Hence

$$\begin{aligned} a(v,t_1) + a(\phi_{t_1}^{\mathrm{AdS}}(v), t_2) &\le d_{\widetilde{\Sigma}}(f(p), z_1) + K + d_{\widetilde{\Sigma}}(z_1,z_2) + 2K \\ &\le d_{\widetilde{\Sigma}}(f(p), z_2) + 3K \\ &= a(v, t_1+t_2) + 4K. \end{aligned}$$

□

Corollary 6.14. *For all* $R > 4K_1^3$ *there exists* $\epsilon > 0$ *such that for all subset* $A \subset C(\Lambda)$*, there exists a covering of* A *by AdS balls* $B_{\mathrm{AdS}}(x_i, R)$*,* x_i *in* A*, and such that* $B_{\widetilde{\Sigma}}(f(x_i),\epsilon) \cap B_{\widetilde{\Sigma}}(f(x_j),\epsilon) = \emptyset$ *for all* $i \neq j$*.*

Proof. Let $A \subset B_{\mathrm{AdS}}(x_i, R)$ be a covering such that $d_{\mathrm{AdS}}(x_i, x_j) > R$ for all $i \neq j$ (to produce such a covering take by induction $x_{n+1} \in A \setminus \cup_{i_1}^n B_{\mathrm{AdS}}(x_i, R)$.) Let $\epsilon \in (0, R/4K_1 - K_1^2)$. Let $z \in B_{\widetilde{\Sigma}}(f(x_i),\epsilon) \cap B_{\widetilde{\Sigma}}(f(x_j),\epsilon)$. Then by Proposition 6.8 we have

$$\begin{aligned} d_{\mathrm{AdS}}(x_i, x_j) &\le K_1 d_{\widetilde{\Sigma}}(f(x_i), f(x_j)) + K_1^2 \\ &\le 2\epsilon K_1 + K_1^2 \\ &\le R/2 \end{aligned}$$

Hence $x_i = x_j$. □

Lemma 6.15.

$$\delta(\Gamma) = \lim_{R\to\infty} \frac{1}{R} \log \mathrm{Vol}(B_{\mathrm{AdS}^3}(o,R) \cap C(\Lambda)).$$

Proof. It is a consequence of the cocompactness of the action on $C(\Lambda)$. It is sufficient to cover $B_{\mathrm{AdS}^3}(o,R) \cap C(\Lambda)$ with translates of a compact fundamental domain and then taking the limit. □

Since the proof consists in comparing the Patterson-Sullivan measures on Λ associated to AdS and to $\widetilde{\Sigma}$ we will name these two measures ν^{AdS} and $\nu^{\widetilde{\Sigma}}$ respectively. Similarly, the two Gromov distances on Λ associated to AdS and $\widetilde{\Sigma}$ distances will be denoted by $d_p^{\mathrm{AdS}}(\cdot,\cdot)$ and $d_p^{\widetilde{\Sigma}}(\cdot,\cdot)$ for $p \in \mathbb{C}(\Lambda)$ or $p \in \widetilde{\Sigma}$ depending on the context.

Lemma 6.16. *For all $p \in \mathrm{AdS}$ and for ν_p^{AdS}-a.e. $\xi \in \Lambda$,*

$$\lim_{t\to\infty} \frac{a(v_p(\xi),t)}{t} = I_m(\widetilde{\Sigma}).$$

Proof. This is a consequence of the product structure of m. We identify $N^1C(\Lambda)$ with $\Lambda^{(2)} \times \mathbb{R}$. Since $a(v,t)$ is Γ invariant, we see that for $\frac{d\nu_p^{\mathrm{AdS}}(\xi)d\nu_p^{\mathrm{AdS}}(\eta)dt}{d_p^{\mathrm{AdS}}(\xi,\eta)}$ a.e. (ξ,η,t) the limit $\lim \frac{a(v_p(\xi),t)}{t}$ exists and is equal to $I_m(\widetilde{\Sigma})$. Here $v_p(\xi)$ is the vector directing (η,ξ), based on p. Clearly, if $\frac{a(v_p(\xi),t)}{t}$ admits a limit then $\frac{a(\phi_{t_0}^{\mathrm{AdS}}v_p(\xi),t)}{t}$ admits the same limit. Hence for $\frac{d\mu_p^{\mathrm{AdS}}(\xi)d\mu_p^{\mathrm{AdS}}(\eta)}{d_p^{\mathrm{AdS}}(\xi,\eta)}$ a.e. (ξ,η) the limit $\lim \frac{a(v(\xi),t)}{t}$ exists and is equal to $I_m(\widetilde{\Sigma})$. Here $v(\xi)$ is any vector directing (η,ξ).

By Lemma 3.6 if $v(\xi) \in (\eta,\xi)$ satisfies $\lim \frac{a(v(\xi),t)}{t} = I_m(\widetilde{\Sigma})$ then $v'(\xi) \in (\eta',\xi)$ satisfies $\lim \frac{a(v'(\xi),t)}{t} = I_m(\widetilde{\Sigma})$ for *every* $\eta' \in \Lambda \setminus \xi$

Hence for ν_p^{AdS}-a.e. $\xi \in \Lambda$,

$$\lim_{t\to\infty} \frac{a(v_p(\xi),t)}{t} = I_m(\widetilde{\Sigma}).$$

□

Inequality case of Theorem 6.5. Choose p as in the previous lemma. For all $\kappa > 0$ and $T > 0$ we define the set

$$A_p^{T,\kappa} = \left\{ \xi \in \Lambda | \left| \frac{a(v_p(\xi),t)}{t} - I_m(\Sigma) \right| \leq \kappa, \quad t \geq T \right\}.$$

For all $d \in]0,1[$ and all $\kappa > 0$, there exists $T > 0$ such that $\nu_p^{\mathrm{AdS}}(A_p^{T,\kappa}) \geq d$. Let $c_{p,\xi}(t) = \pi(\phi_t^{\mathrm{AdS}}(v_p(\xi)))$ be the geodesic of AdS. For $t > T$ consider the subset $\{c_{p,\xi}(t) | \xi \in A_p^{T,\kappa}\} \subset S(p,t)$ of the Lorentzian sphere of radius t and center p in AdS.

Choose a covering of this subset by balls $B_{\mathrm{AdS}}(x_i,R)$ with R sufficiently large such that corollary 6.14 and the shadow lemma for ν_p^{AdS} apply. Then, by the local behaviour of ν_p^{AdS}, there exists a constant $c > 1$ independent of t such that

$$\frac{1}{c}e^{-\delta t} \leq \nu_p(\mathcal{S}_{\mathrm{AdS}}(x_i,R))) \leq ce^{-\delta t}.$$

It is clear by definition that $A_p^{T,\epsilon} \subset \cup_{i\in I}\mathcal{S}_{\mathrm{AdS}}(x_i,R))$ and therefore,

$$d \leq \nu_p^{\mathrm{AdS}}\left(\cup_{i\in I}\mathcal{S}_{\mathrm{AdS}}(x_i,R))\right) \leq \sum_{i\in I}\nu_p^{\mathrm{AdS}}(\mathcal{S}_{\mathrm{AdS}}(x_i,R))) \leq c\,\mathrm{Card}(I)e^{-\delta t}. \quad (7)$$

By the property of the covering of 6.14 the balls $\left\{B_{\widetilde{\Sigma}}(f(x_i),\epsilon)\right\}_{i\in I}$ are disjoint. Moreover $d_{\widetilde{\Sigma}}(f(p),f(x_i)) \leq t(I_m(\widetilde{\Sigma})+\kappa))$ by definition of $A_p^{T,\kappa}$. Hence the balls $B_{\widetilde{\Sigma}}(f(p),t(I_m(\widetilde{\Sigma})+\kappa)+\epsilon)$ contains the disjoint union $\sqcup_{i\in I}B_{\widetilde{\Sigma}}(f(x_i),\epsilon)$. Let $v := \min_{i\in I}\mathrm{Vol}\,B_{St}(f(x_i),\epsilon)$. Then

$$\mathrm{Vol}\,B_{\widetilde{\Sigma}}(f(p),t(I_m(\widetilde{\Sigma})+\kappa)+\epsilon) \geq v\,\mathrm{Card}(I). \quad (8)$$

By equations 7 and 8, we have

$$\begin{aligned} e^{\delta t} &\leq \frac{c}{d}\mathrm{Card}(I) \\ &\leq \frac{c}{vd}\mathrm{Vol}\,B_{\widetilde{\Sigma}}(f(p),t(I_m(\widetilde{\Sigma})+\kappa)+\epsilon) \end{aligned}$$

We conclude using Lemma 6.15 since κ is arbitrary. □

6.3 Proof of rigidity

We will need the following property of cocycle proved by G. Knieper.

Lemma 6.17. *[Kni95] There exists a constant L such that for ν_p^{AdS} almost all $\xi \in \Lambda$ there is a sequence $t_n \to \infty$ such that*

$$|d_{\widetilde{\Sigma}}(f(p),f(\pi\phi_{t_n}^{\mathrm{AdS}}v_p(\xi))) - I_m(\widetilde{\Sigma})t_n| \leq L.$$

As it is explained in [Kni95] it is a consequence of Corollary 6.13 that for m a.e. $v \in N^1C(\Lambda)/\Gamma$ there is a subsequence t_n such that $|a(v,t_n) - I_m(\widetilde{\Sigma})t_n| \leq L$. As in Lemma 6.16 we pass from m a.e. $v \in N^1C(\Lambda)/\Gamma$ to ν_p^{AdS} a.e. $\xi \in \Lambda$

Proposition 6.18. *If there is equality in Eq.(6), then the Patterson Sullivan measures associated to $\widetilde{\Sigma}$ and* AdS *are equivalent.*

Proof. Let $\xi \in \Lambda$ be a generic point for ν_p^{AdS} and set $y_n := \pi\phi_{t_n}^{\mathrm{AdS}}v_p(\xi)$. From the previous lemma we have

$$|d_{\widetilde{\Sigma}}(f(p),f(y_n)) - I_m(\widetilde{\Sigma})t_n| \leq L. \quad (9)$$

Let R be large enough for the shadow lemma hold in both $\widetilde{\Sigma}$ and AdS. According to 6.11 there exists $R', R'' > R$ such that for all $x \in \widetilde{\Sigma}$ we have :

$$\mathcal{S}_{\mathrm{AdS}}(x,R) \subset \mathcal{S}_{\widetilde{\Sigma}}(x,R') \subset \mathcal{S}_{\mathrm{AdS}}(x,R'').$$

Applying ν_{AdS} with $x = f(y_n)$ we get :

$$\nu_{\mathrm{AdS}}(\mathcal{S}_{\mathrm{AdS}}(f(y_n), R)) \leq \nu_{\mathrm{AdS}}(\mathcal{S}_{\widetilde{\Sigma}}(f(y_n), R')) \leq \nu_{\mathrm{AdS}}(\mathcal{S}_{\mathrm{AdS}}(f(y_n), R'')).$$

By the local behaviour of ν_{AdS} there exists $c > 1$ such that

$$\frac{1}{c} e^{-\delta d_{\mathrm{AdS}}(f(p), f(y_n))} \leq \nu_{\mathrm{AdS}}(\mathcal{S}_{\widetilde{\Sigma}}(f(y_n), R')) \leq c e^{-\delta d_{\mathrm{AdS}}(f(p), f(y_n))}.$$

By the local behaviour of $\nu_{\widetilde{\Sigma}}$ there exists $c_2 > 1$ such that

$$\frac{1}{c_2} e^{-h(\widetilde{\Sigma}) d_{\widetilde{\Sigma}}(f(p) f(, y_n))} \leq \nu_{\widetilde{\Sigma}}(\mathcal{S}_{\widetilde{\Sigma}}(f(y_n), R')) \leq c_2 e^{-h(\widetilde{\Sigma}) d_{\widetilde{\Sigma}}(f(p), f(y_n))}.$$

From equation 9, there exist $c_3 > 1$ such that

$$\frac{1}{c_3} \leq \frac{e^{-h(\widetilde{\Sigma}) d_{\widetilde{\Sigma}}(f(p) f(, y_n))}}{e^{-\delta d_{\mathrm{AdS}}(f(p), f(y_n))}} \leq c_3$$

Hence there exists $c_4 > 1$ such that

$$\frac{1}{c_4} \leq \frac{\nu_{\mathrm{AdS}}(\mathcal{S}_{\widetilde{\Sigma}}(f(y_n), R'))}{\nu_{\widetilde{\Sigma}}(\mathcal{S}_{\widetilde{\Sigma}}(f(y_n), R'))} \leq c_4.$$

Since $\mathcal{S}_{\widetilde{\Sigma}}(f(y_n), R')) \to_{n \to \infty} \xi$, this concludes the proposition. □

Proposition 6.19. *If the Patterson-Sullivan measures ν_{AdS} and ν_{Σ} are equivalent then the Gromov distances on Λ seen as $\partial C(\Lambda) \cap \partial \mathrm{AdS}$ or $\partial \widetilde{\Sigma}$ are Hölder équivalent.*

Proof. Consider on $\Lambda^{(2)}$ the Bowen-Margulis currents defined by

$$\mu_{\widetilde{\Sigma}}(\xi, \eta) = \frac{d\nu^p_{\widetilde{\Sigma}}(\xi) d\nu^p_{\widetilde{\Sigma}}(\eta)}{d_p^{\widetilde{\Sigma}}(\xi, \eta)^{2h}}$$

$$\mu_{\mathrm{AdS}}(\xi, \eta) = \frac{d\nu^p_{\mathrm{AdS}}(\xi) d\nu^p_{\mathrm{AdS}}(\eta)}{d_p^{\mathrm{AdS}}(\xi, \eta)^{2\delta}}.$$

By assumption $\nu^p_{\widetilde{\Sigma}}$ and ν_p^{AdS} are equivalent, therefore $\mu_{\widetilde{\Sigma}}$ and μ_{AdS} are also equivalent. The ergodicity and the Γ-invariance implies the existence of $c > 0$ such that

$$\mu_{\widetilde{\Sigma}} = c \mu_{\mathrm{AdS}}.$$

Since $\nu_p^{\widetilde{\Sigma}}$ and ν_p^{AdS} are equivalent there exists a function $u : \Lambda \to \mathbb{R}^+$ such that $\nu_p^{\widetilde{\Sigma}}(\xi) = u(\xi) \nu_p^{\mathrm{AdS}}(\xi)$. We have

$$u(\xi) u(\eta) d_p^{\mathrm{AdS}}(\xi, \eta)^{\delta} = c d_p^{\widetilde{\Sigma}}(\xi, \eta)^{h}.$$

We see that u is equal almost everywhere to a continuous function. We can therefore suppose that u is continuous on Λ hence strictly positive. By compacity, there exists $C > 1$ such that $\frac{1}{C} \leq u(\xi) \leq C$. Finally we get what we stated

$$\frac{c}{C^2} d_p^{\widetilde{\Sigma}}(\xi,\eta)^h \leq d_p^{\mathrm{AdS}}(\xi,\eta)^\delta \leq C^2 c d_p^{\widetilde{\Sigma}}(\xi,\eta)^h.$$

□

Proposition 6.20. *If the two Gromov distances coming from the distance on $\widetilde{\Sigma}$ and on $C(\Lambda)$ are Hölder equivalent then the marked length spectra of $M = E(\Lambda)/\Gamma$ and $\widetilde{\Sigma}/\Gamma$ are proportional.*

Proof. Let $g \in \Gamma$ and $\xi \in \Lambda \setminus \{g^\pm\}$.Then

$$[g^-, g^+, g(\xi), \xi]_{\mathrm{AdS}} = e^{\ell_{\mathrm{AdS}}(g)}$$

and

$$[g^-, g^+, g(\xi), \xi]_\Sigma = e^{\ell_\Sigma(g)}$$

By assumption on the Gromov distances $d_{\widetilde{\Sigma}}, d_{\mathrm{AdS}}$ and from the definition of the cross ratio, Eq. (1), there exists $C > 1$ such that for all $g \in \Gamma$ we have

$$\frac{1}{C} e^{r\ell_{\mathrm{AdS}}(g)} \leq e^{\ell_\Sigma(g)} \leq C e^{r\ell_{\mathrm{AdS}}(g)}.$$

In particular when we look at the power g^n of g, taking the log, we get for all $n > 0$ and all $g \in \Gamma$:

$$-\log(C) + rn\ell_{\mathrm{AdS}}(g) \leq n\ell_\Sigma(g) \leq \log(C) + rn\ell_{\mathrm{AdS}}(g).$$

We finish the proof by dividing by n and taking the limit :

$$\ell_\Sigma(g) = r\ell_{\mathrm{AdS}}(g).$$

□

Putting everything together we obtain the equivalent of Bowen's Theorem for AdS quasi-Fuchsian manifolds

Corollary 6.21. $\delta(\Gamma) \leq 1$ *with equality if and only if* Γ *is Fuchsian.*

Proof. The boundary of the convex core is isometric to $\mathbb{H}^2$ hence its volume entropy is 1. Applying Theorem 6.5 and Proposition 6.9, to these surfaces we have $\delta \leq 1$ with equality if and only if the boundary of the convex core has the same length spectrum as $M = E(\Lambda)/\Gamma$. Let M be parametrized in the Mess' parametrization by S_1 and S_2. Recall that for a homotopy class of closed curved c, the geodesic length of c in M is $\ell_{Lor}(c) = \frac{\ell_1(c)+\ell_2(c)}{2}$ see [Glo15a]. Then, corollary 6.21 is a consequence of the following hyperbolic geometry result showing that $S_1 = S_2$. □

Lemma 6.22. *Let S_1, S_2, S_3 be three hyperbolic surfaces. Let ℓ_j be their corresponding length functions. If for all closed homotopy classes of closed curves c we have $\ell_1(c) + \ell_2(c) = 2\ell_3(c)$ then $S_1 = S_2$ $(= S_3)$.*

Proof. Recall from Bonahon, that the function ℓ_j extends continuously on the set of geodesic currents. They are the restriction of the intersection function with the Liouville current L_j : for all $c \in \mathcal{C}$ we have $\ell_j(c) = i(c, L_j)$, [Bon88, Proposition 14]. Moreover, the set of closed geodesics is dense in the set of currents [Bon88, Proposition 2]. Hence by applying to L_3 the condition on the length functions we get

$$i(L_1, L_3) + i(L_2, L_3) = 2i(L_3, L_3).$$

Here again, we use Bonahon result saying that for any two Liouville currents L, L' we have $i(L, L') \geq i(L, L)$ with equality iff $L = L'$, [Bon88, Theorem 19]. It implies that $L_1 = L_3$ and $L_2 = L_3$. By Bonahon's [Bon88, Lemma 9] we have

$$S_1 = S_2 = S_3.$$

□

7 Appendix

The proof of ergodicity of a conformal density ν_x uses the existence of Lebesgue density points. The proof of Lebesgue differentiating Theorem follows from classical analysis once one has a Hardy-Littlewood's type inequality for maximal function over the subsets of shadows. The latter inequality follows from Vitali's covering lemma.

Lemma 7.1 (Vitali)**.** *Let $G \subset \Gamma$. Consider the union $\cup_{g \in G} \mathcal{S}_R(x, gx)$, there exists a subset $G' \subset G$ such that*

- $\mathcal{S}_R(x, g'x)$ *are pairwise disjoint for* $g' \in G'$.
- $\cup_{g \in G} \mathcal{S}_R(x, gx) \cap \Lambda \subset \cup_{g' \in G'} S_{5R+5k}(x, g'x)$.

Proof. Let $G = \{g_i\}_{i \in \mathbb{N}}$ be such that $d(x, g_i x) \leq d(x, g_{i+1} x)$. Define by induction $i_0 = 0$ and

$$i_{k+1} = \min\{i > i_k \,|\, \mathcal{S}_R(x, g_i x) \cap \cup_{j \leq k} \mathcal{S}_R(x, g_{i_j} x) = \emptyset\}.$$

If $i_k < i < i_{k+1}$ for some k, then $\mathcal{S}_R(x, g_i x) \cap \mathcal{S}_R(x, g_{i_j} x) \neq \emptyset$ for a $j \leq k$. Hence there is a geodesic ray from x which crosses $B(g_{i_j} x, R)$ and $B(g_i x, R)$. We call $\xi_1 \in \partial\mathrm{AdS}$ the limit point of this geodesic ray. Let $\xi_2 \in \mathrm{AdS}$ be another element of $\mathcal{S}_R(x, g_i x)$. We parametrize $[x, \xi_1)$ and $[x, \xi_2)$ by

$$f_1(t) = \cosh(t)x + \sinh(t)\,(\xi_1 - x)\,.$$

$$f_2(t') = \cosh(t')x + \sinh(t')\,(\xi_2 - x)\,.$$

The distance between $f_1(t)$ and the ray $[x,\xi_2)$ is given by $d(f_1(t),[x,\xi_2)) = \inf_{t'>0} d(f_1(t), f_2(t'))$. By an explicit computation, we find

$$|\langle f_1(t)\,|\,f_2(t')\rangle| = |e^{t'}A(t) + e^{-t'}B(t)|$$

where $A(t) = \left(\langle x\,|\,\xi_2\rangle \frac{e^{-t}}{2} + \langle x\,|\,\xi_1\rangle \sinh(t) + \langle \xi_1\,|\,\xi_2\rangle \frac{\sinh(t)}{2}\right)$
and $B(t) = \left(-e^{-t} - \langle x\,|\,\xi_2\rangle \frac{e^{-t}}{2} + \langle x\,|\,\xi_1\rangle \sinh(t) - \langle \xi_1\,|\,\xi_2\rangle \frac{\sinh(t)}{2}\right)$.

Depending on the sign of $\langle x\,|\,\xi_1\rangle + \frac{\langle \xi_1\,|\,\xi_2\rangle}{2}$, the function $t' \to |\langle f_1(t)\,|\,f_2(t')\rangle|$ is bounded or increasing for t sufficiently large. This implies that the distance is bounded or increasing.

Let $f_1(t_i) = z_i$ and $f_1(t_{i_j}) = z_{i_j}$ be two points on $[x,\xi_1)$ such that $z_i \in B(g_i, R)$ and $z_{i_j} \in B(g_{i_j}, R)$.

Suppose first that $t_{i_j} < t_i$. Let z_i^2 be a point in $[x,\xi_2)$ such that $d(z_i^2, g_i x) \leq R$. This implies that $d(z_i^2, z_i) \leq d(z_i^2, g_i x) + d(g_i x, z_i) + k \leq 2R + k$. Then by the previous analysis there exists $z_{i_j}^2$ on $[x,\xi_2)$ such that $d(z_{i_j}^2, z_{i_j}) \leq 2R + k$. We then have

$$\begin{aligned} d(g_{i_j}x, [x,\xi_2)) &\leq d(g_{i_j}x, z_{i_j}^2) \\ &\leq d(g_{i_j}x, z_{i_j}) + d(z_{i_j}^2, z_{i_j}) + k \\ &\leq 3R + 2k. \end{aligned}$$

In other words

$$\mathcal{S}_R(x, g_i x) \subset S_{3R+2k}(x, g_{i_j}x).$$

Suppose now that $t_{i_j} \geq t_i$. Since $f_1(t_i) = z_i$ and $f_1(t_{i_j}) = z_{i_j}$ are on the same space like geodesic we have

$$d(z_{i_j}, z_i) = d(x, z_{i_j}) - d(x, z_i). \tag{10}$$

Moreover

$$\left\{ \begin{array}{lcl} d(x, z_{i_j}) & \leq & d(x, g_{i_j}x) + d(g_{i_j}x, z_{i_j}) + k \\ d(x, g_i x) & \leq & d(x, z_i) + d(z_i,, g_i x) + k \end{array} \right.$$

$$\left\{ \begin{array}{lcl} d(x, z_{i_j}) & \leq & d(x, g_{i_j}x) + R + k \\ -d(x, z_i) & \leq & -d(x, g_i x) + R + k \end{array} \right.$$

Then (10) becomes

$$d(z_{i_j}, z_i) \leq d(x, z_{i_j}) - d(x, z_i) + 2R + 2k \leq 2R + 2k$$

as $d(x, z_{i_j}) \leq d(x, z_i)$.

We finally have for all $z \in B(g_i x, R)$

$$\begin{aligned} d(z, g_{i_j}x) &\leq d(z, g_i x) + d(g_i x, z_i) + d(z_i, z_{i_j}) + d(z_{i_j}, g_{i_j}x) + 3k \\ &\leq 5R + 5k \end{aligned}$$

Or in other words :

$$\mathcal{S}_R(x, g_i x) \subset S_{5R+5k}(x, g_{i_j} x).$$

□

Let $\phi \in L^1(\nu_x)$ and define the maximal function associated to ϕ by

$$\phi^*(\xi) = \lim_{t\to\infty} \sup \frac{1}{\nu_x(\mathcal{S}_R(x,\gamma x))} \int_{\nu_x(\mathcal{S}_R(x,\gamma x)} \phi d\nu_x$$

where the sup is taken over all the $\gamma \in \Gamma$ such that $d(\gamma x, x) \geq t$ and $\xi \in \mathcal{S}_R(x, \gamma x)$.

Lemma 7.2 (Hardy-Littlewood inequality)**.** *There exist $C > 0$ such that for all ϕ and all $\epsilon > 0$*

$$\nu_x(\{\phi* > \epsilon\}) \leq \frac{C}{\epsilon}\|\phi\|_{\nu_0}.$$

Proof. Fix $\epsilon > 0$. We have by definition of the maximal function a subset $G \in \Gamma$ such that

$$\{\phi* > \epsilon\} \subset \cup_{g\in G}\mathcal{S}_R(x, gx)$$

and

$$\frac{1}{\nu_x(\mathcal{S}_R(x, gx))} \int_{\nu_x(\mathcal{S}_R(x,gx)} \phi d\nu_x > \epsilon.$$

By Vitalli's covering Lemma, there is $G' \subset G$ such that $\mathcal{S}_R(x, g'x)$ are pairwise disjoint for $g' \in G'$ and $\{\phi* > \epsilon\} \subset \cup_{g'\in G'} S_{5R+5k}(x, g'x)$. Choose R big enough, such that Shadow Lemma Theorem 1.2 applies. There exist C such that

$$\nu_x(\mathcal{S}_{5R+5k}(x, g'x)) \leq C\nu_x(\mathcal{S}_R(x, g'x))$$

where C is independent of γ and ϕ. We finally have

$$\begin{aligned}
\nu_x(\{\phi* > \epsilon\}) &\leq \sum_{g\in G'} \nu_x(\mathcal{S}_{5R+5k}(x, g'x)) \\
&\leq C \sum_{g\in G'} \nu_x(\mathcal{S}_R(x, g'x)) \\
&\leq \frac{C}{\epsilon} \sum_{g'\in G'} \int_{\nu_x(\mathcal{S}_R(x,gx)} \phi d\nu_x \\
&\leq \frac{C}{\epsilon} \int_\Lambda |\phi| d\nu_x
\end{aligned}$$

□

By a usual argument of density of continuous function in $L^1(\nu_x)$ we have

Lemma 7.3 (Lebesgue differentiating theorem)**.** *For R sufficiently large we have for all $f \in L^1(\nu_x)$*

$$\lim_{n\to\infty} \frac{1}{\nu_x(\mathcal{S}_R(x, g_n x))} \int_{\mathcal{S}_R(x,g_n x)} f d\nu_x = f(\xi),$$

as $d(x, g_n x) \to \infty$ and $\xi \in \mathcal{S}_R(x, g_n x)$.

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